TURN ON MICROPHONE

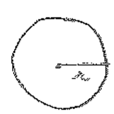
Welcome to in-class & online students
Hand out syllabus & 1st HW assignment
Point out e-mail address
Change in o.o. — held in OH 109.
Registratn #'s — have 8 online, 5 in-class (need 5 to keep course)

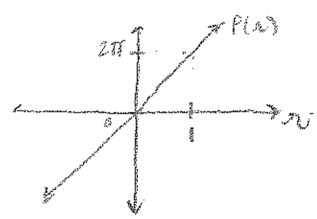
Begin — a review of fns., models, graphs.

Def. A real-valued function (fn.) f defined on a set $D \subseteq \mathbb{R}$ ("domain", subset, reals) is a rule that assigns to each input $x \in D$ ("element of") exactly one output real # $f(x) \in R$ ("range")

"indep. var." "dependent var."

Notes. • D may be restricted — depends on modelling goal. this may affect the range.

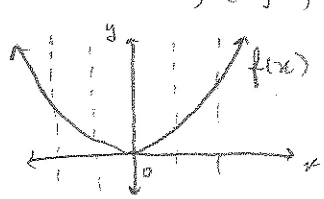
e.g.,  $P(r) = 2\pi r$



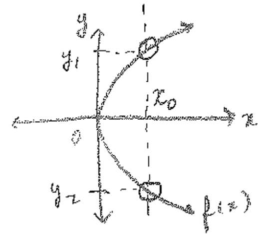
As a mathematical construct, $D = \mathbb{R} = R$
But practically, $r<0$ makes no sense. So restrict

$$D = \underbrace{[0, +\infty)}_{\text{interval notation}} = \underbrace{\{x \in \mathbb{R} : x \geq 0\}}_{\text{set notation}}, \text{ and } R = D.$$

• For functions, "exactly one" is key — recall "vertical line test", e.g.,



is but



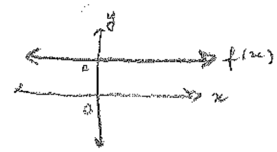
is not : because what's $f(x_0)$? y_1 or y_2??

some functions & examples...

① Polynomials: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 (\overset{1}{\overset{\shortparallel}{x^0}})$

n is "degree" $(a_n \neq 0)$

- Constants (deg. 0) $f(x) = c$



$D = \mathbb{R} = (-\infty, +\infty)$ "singleton"

$R = \{c\} = [c, c]$ → set notation $\{\ \}$

- Linear (deg. 1) $y = mx + b$ "slope intercept"

b: "y-intercept" – when $x = 0$, $y = b$, i.e., $(0, b)$ on the line

slope $= \dfrac{\text{"rise"}}{\text{run}} = \dfrac{\Delta y}{\Delta x}$

Δ: capital Delta = "change in"

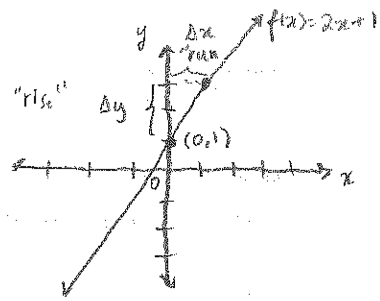
(x_0, y_0) and (x_1, y_1)

$\dfrac{y_1 - y_0}{x_1 - x_0} = m$ $(y - y_0) = m(x - x_0)$ "point-slope"

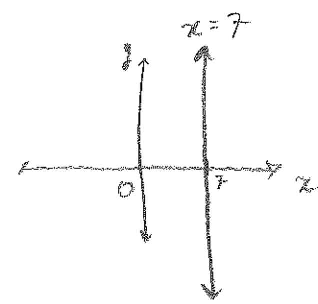
"when $x = x_0$, $y = y_0$" i.e., (x_0, y_0) on the line

e.g., $y = \overset{\frac{2}{1}}{\overset{\shortparallel}{2}}x + 1$

$D = R = \mathbb{R} = (-\infty, +\infty)$



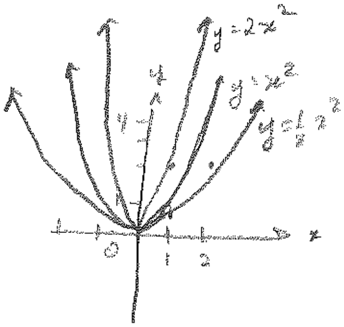
Vertical lines aren't functions!



What's $f(7)$??
(fails v.l.t.)

- QUADRATIC
  DEG. 2

$y = ax^2 + bx + c$



$\downarrow$ "completing the square": $\frac{y}{a} = x^2 + \frac{b}{a}x + \frac{c}{a} = (x + \frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{2a^2}$

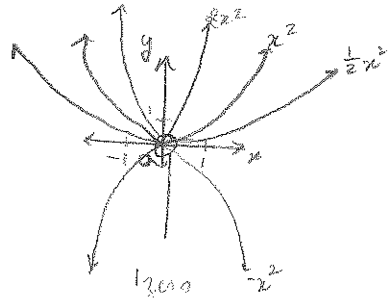
$\boxed{y - k = a(x-h)^2}$

$\Rightarrow y - \boxed{\underbrace{\left(c - \frac{b^2}{2a}\right)}_{k}} = a(x - \boxed{\underbrace{\frac{-b}{2a}}_{h}})^2$

a: • "width" of parabola: big a, skinny parabola
small a, wide parabola
• "direc'n": a<0 - open down
>0 - up

$\rightarrow (h,k)$: vertex

e.g.,



D = $\mathbb{R}$

R = $\{x \in \mathbb{R} : x \geq 0\}$ = $[0, +\infty)$

set of x in $\mathbb{R}$ such that

closed — "contains boundary"

open (always open at $\pm\infty$ for now)

$ax^2 + bx + c = 0$

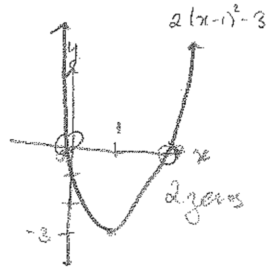
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

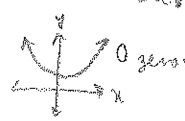
$\sqrt{-4}$

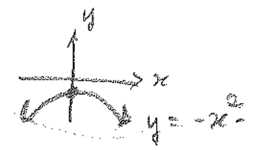
$y + 3 = 2(x-1)^2$

vertex: $(1,-3)$

a = 2

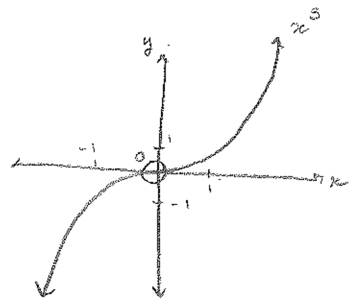




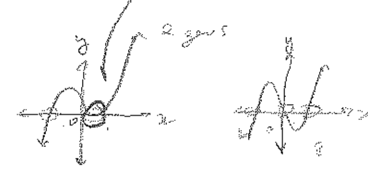


- CUBIC
  deg. 3

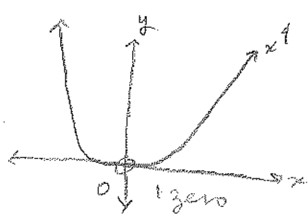
$y = ax^3 + bx^2 + cx + d$



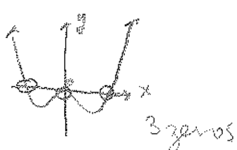
$y = (x+7)(x-1)^2$

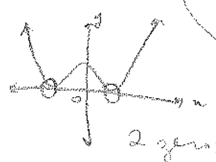


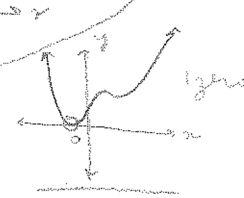
- QUARTIC
  deg 4



"W-shaped"





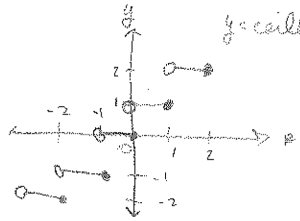


In general... • odd (nonzero) deg. polys have at least one zero
• even may not have any?

look at limiting behavior!
$\longrightarrow$ more precise later, but — H.O.T. (higher order terms) grow fastest ("dominate" as $x \to \pm\infty$), so for odd, $f \xrightarrow[x\to\pm\infty]{} \pm\infty$, even $f \xrightarrow[x\to\pm\infty]{} +\infty$.

- Step functions

e.g., ceil(x)

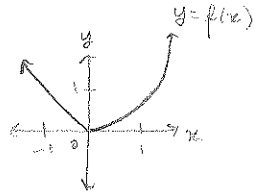


$D = \mathbb{R}$

$R = \mathbb{Z}$ (integers)

- Piecewise fns

e.g., $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x, & x < 0 \end{cases}$



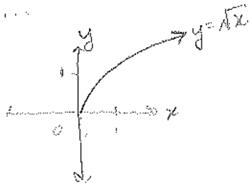
$D = \mathbb{R}$

$R = [0, +\infty)$

Other functions...

$f(x) = \sqrt{x}$

"algebraic"

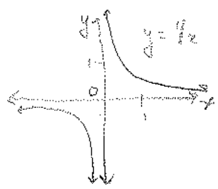


$D = [0, +\infty) = \{x \in \mathbb{R} : x \geq 0\}$

$R = [0, +\infty) = \{x \in \mathbb{R} : x \geq 0\}$

$g(x) = \frac{1}{x}$

"rational"



singleton — "union"

$D = \mathbb{R} \setminus \{0\} = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, +\infty)$

$R = D$ — set minus ("excluding")

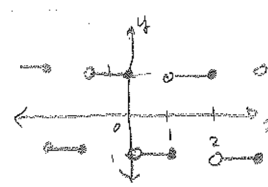
ex. Find domain of $f(x) = \frac{1}{\sqrt{2x-4}}$.

For sqrt to exist, want $2x - 4 \geq 0 \Rightarrow x \geq 2$.

For denom. to be nonzero, want $\sqrt{2x-4} \neq 0 \Rightarrow x \neq 2$

Taking both together, ~~want~~ have $D = (2, +\infty) = \{x \in \mathbb{R} : x > 2\}$

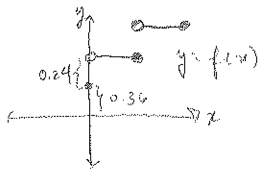
Ex. find range: $f(x) = (-1)^{\text{ceil}(x)} = \begin{cases} 1, & \text{ceil}(x) \text{ even} \\ -1, & \text{ceil}(x) \text{ odd} \end{cases}$



so $R = \{-1, 1\}$ $[-1] \cup [1]$ $D = \mathbb{R}$

— , — letter is 0.39 to send, plus 0.24/add'l ounce or frac. thereof

so $f(x) = 0.39 + 0.24\,\text{ceil}(x)$



so $D: \{x \in \mathbb{R} : x = 0.39 + 0.24m,\ m \in \mathbb{N}\}$

natural #'s: 1, 2, 3, ...

Rational fns, e.g. $f(x) = \dfrac{(x+2)(x-1)}{x(x+1)(x-2)}$ → quadratic / → cubic

How would we possibly graph this?

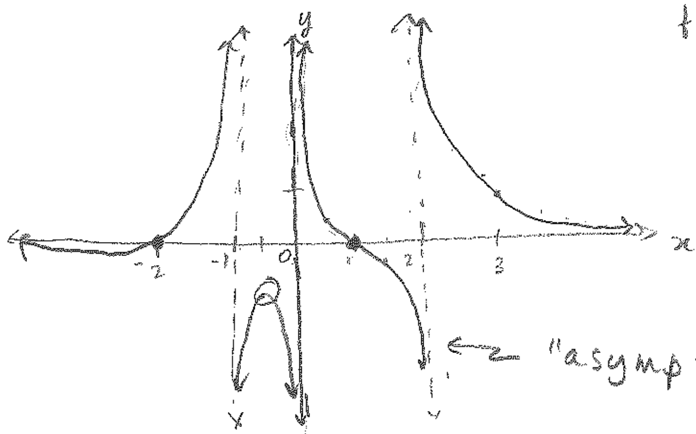
- Undefined when $x=0$, $x=-1$, or $x=2$.
- Zero when $x=-2$ or $x=1$.
- Check test points elsewhere... $f(3) = \dfrac{5\cancel{(2)}}{3\cancel{(4)}\cancel{(1)}2} = \dfrac{5}{6}$

$f(1.5) = \dfrac{(3.5)\cancel{(0.5)}}{1.5(2.5)\cancel{(-0.5)}} < 0$

$f(0.5) = \dfrac{2.5\cancel{(-0.5)}}{\cancel{0.5}(1.5)\cancel{(-1.5)}} > 0$

$f(-0.5) = \dfrac{1.5(-1.5)}{-0.5(0.5)(-2.5)} < 0$

See p.30



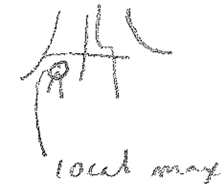
- Check limiting behavior $f(x) \xrightarrow[x \to \pm\infty]{} 0$

These are the tools we have now — but calculus gives us some more — like:

- where is f decreasing / increasing?
- ——— " ——— concave up/down?

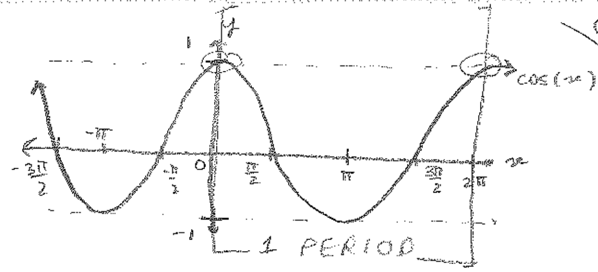
"smiles" - both increasing, one "frowning"

- where are the local extrema?
  ↳ max / min e.g.

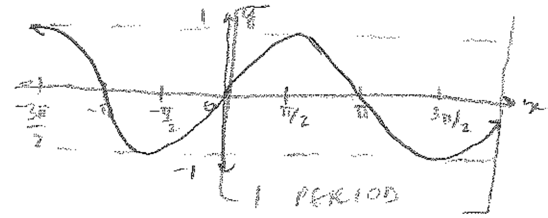


More functions...

- Trigonometric, e.g., $f(x) = \cos(x)$



$f(x) = \sin(x)$

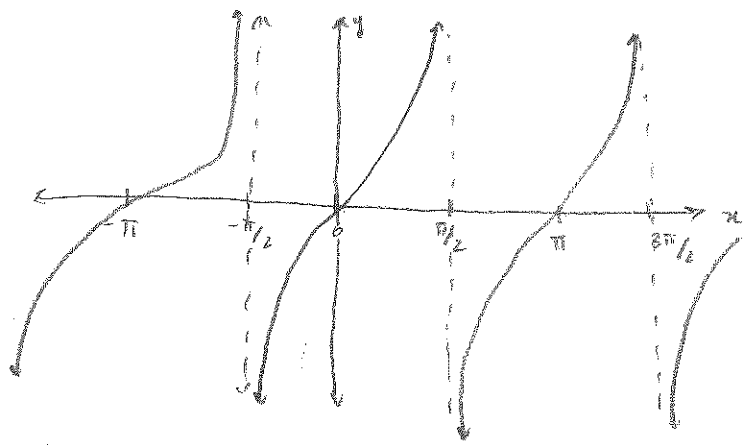


$D = \mathbb{R}$
$R = [-1, 1]$

- Periodicity: $\forall$ (for all) $x \in \mathbb{R}$, $\sin(x + 2\pi) = \sin(x)$ Period $= 2\pi$
$\cos(x + 2\pi) = \cos(x)$

e.g., $f(x) = \tan(x)$

$= \dfrac{\sin(x)}{\cos(x)}$



- Asymptotes when $\cos(x) = 0$, i.e., at $\frac{\pi}{2}$ and $\frac{\pi}{2} + m\, 2\pi$, $m \in \mathbb{Z}$

Set of asymptotes: $\{x \in \mathbb{R} : x = \frac{\pi}{2} + 2\pi m,\ m \in \mathbb{Z}\}$

- Zeros when $\sin(x) = 0$, i.e., at $0, \pi + 2\pi m$, $m \in \mathbb{Z}$

Set of zeros: $\{x \in \mathbb{R} : x = \pi m,\ m \in \mathbb{Z}\}$.

Shifting / changing the graphs of sin / cos.

$$f(x) = \underbrace{A}_{\substack{\text{"amplitude"} \\ = \text{vertical scale}}} \cdot \sin(\underbrace{C}_{\text{horizontal scale}}x - \underbrace{h}_{\text{horizontal shift}}) + \underbrace{k}_{\text{vertical shift}}$$

Period $= \dfrac{2\pi}{C}$

Transformations work this way for other graphs too (remember parabola?)

$$y - k = \underbrace{a}_{\text{vertical scale}} \cdot f(\underbrace{c}_{\text{horiz. scale}}x - h)$$

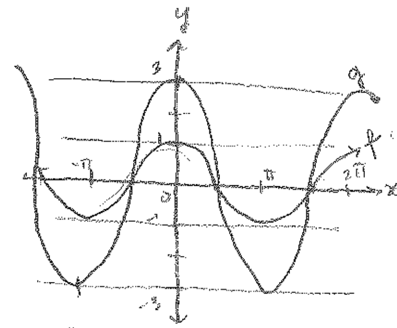
h → horiz. shifts; k → vert. shifts

e.g., $f(x) = \cos(x)$

$g(x) = 3\cos(x)$ is a vertical scaling;

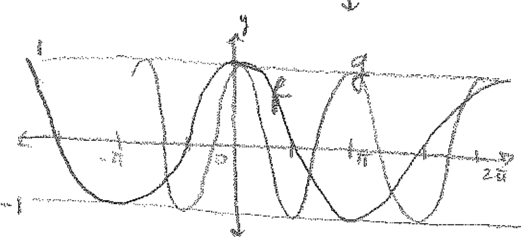
$R = [-1, 1]$

$R = [-3, 3]$

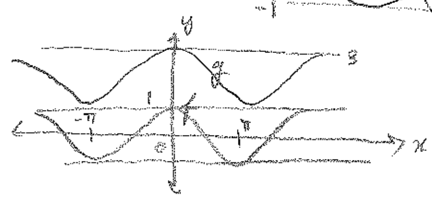


$g(x) = \cos(2x)$ horiz. scaling.

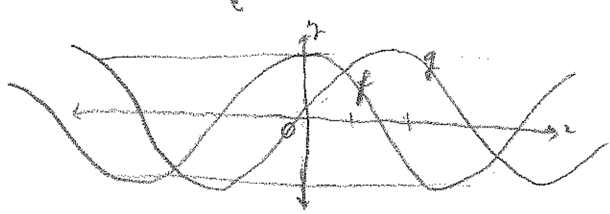
(period $= \frac{2\pi}{2} = \pi$)



$g(x) = \cos(x) + 2$



$g(x) = \cos(x+2)$

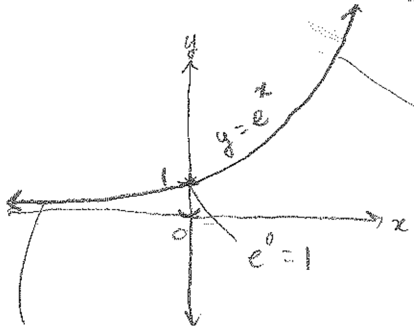


(show on own paper)

Fns, ct'd.

~~$f(x) = x^2$~~

- Exponential $\quad f(x) = \underbrace{b}_{\text{"base"}}{}^{x} \qquad$ e.g., $b = e \approx 2.718$



really fast growth!

$D = \mathbb{R}$

$R = (0, +\infty)$ ↳ open endpoint.

$e^x \xrightarrow[x \to -\infty]{} 0 \quad$ (recall, $e^{-x} = \frac{1}{e^x}$)

$e^x \xrightarrow[x \to +\infty]{} +\infty$

<u>ex</u>. Compounding interest (p 39)

Invest \$ p in an acc't with 5% interest, comp. annually.

End of 1st yr: $\quad f(1) = \underbrace{p(1.05)}$

2nd : $\quad f(2) = \left[p(1.05)\right](1.05) = p(1.05)^2$

3rd : $\quad f(3) = \left[p(1.05)^2\right](1.05) = p(1.05)^3$

$\vdots$

So, at end of n^{th} yr, have $f(n) = p(1.05)^n$

<u>Domain</u>: $\quad D = \mathbb{N}, \qquad \mathbb{R}, \mathbb{Z}, \mathbb{N}$ "natural numbers" $= \{1, 2, 3, 4, 5, \ldots\}$

$R = [p(1.05), +\infty)$

Functions, ct'd.

- Logarithmic

Recall: $\underbrace{b}_{\text{"base", } b>0}^{a} = x \Rightarrow a = \underbrace{\log_b x}_{\text{"log base } b\text{"}}$

common: $\underline{b=10}$ or $b=e$ (natural log; denoted ln)

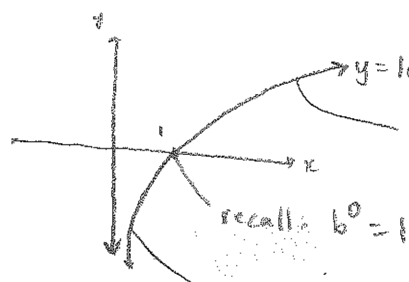
$e^a = x \Rightarrow a = \underline{\ln x}$

What would it mean to take $\log_b(0)$?

$\log_b(0) = a \Rightarrow b^a = 0$. Not true for any a !

So $\log(0)$ undef. Same for neg. #s.

Thus,



$y = \log_b(x)$

growth, but not as fast

recall: $b^0 = 1 \ \forall b$

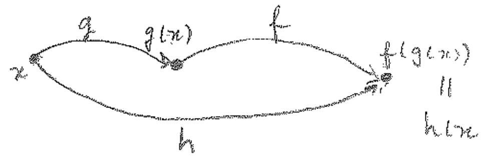
recall: $b^a \xrightarrow[a \to -\infty]{} 0$, so $\log_b x \xrightarrow[x \to 0]{} -\infty$

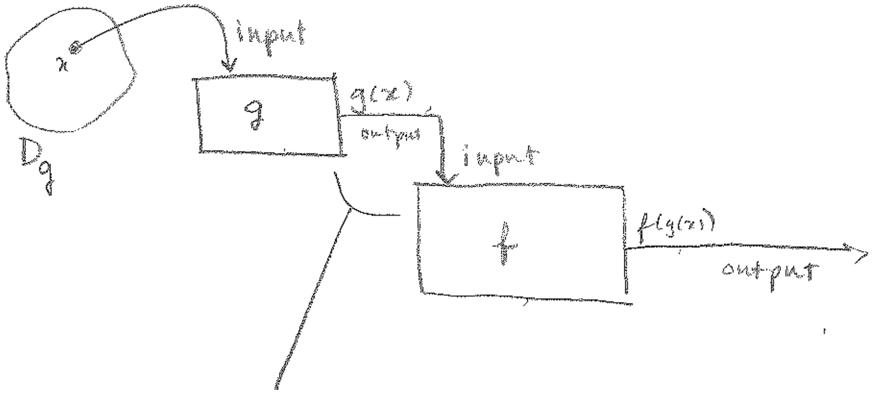
$D = \{x \in \mathbb{R} : x > 0\}$
$= (0, +\infty)$

$R = \mathbb{R}$

Function composition.

Def. $h(x) = f \circ g(x) = \underbrace{f(g(x))}$



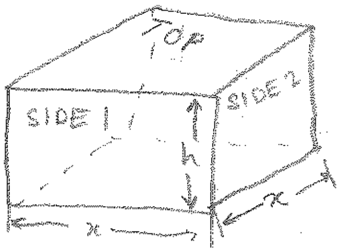


THIS PART ONLY WORKS IF $g(x) \in D_f$ as well !!

Mathematical modelling.

Ex. A rectangular box w/square base has volume $V = 125\ m^3$. Express its total surface area A as a fn. of the edge length x of its base.

8, p.5



Relevant formulas are:

(1) $V = x^2 h = 125\ m^3$

(2) $A = 2(\underbrace{x^2}_{\text{area of top}} + \underbrace{xh}_{\text{area of side 1}} + \underbrace{xh}_{\text{area of side 2}}) = 2x^2 + 4xh$

By (1), have $h = \frac{125\ m^3}{x^2}$ (1*), and plugging into (2):

$$A = 2x^2 + 4x\left(\frac{125\ m^3}{x^2}\right) = 2x^2 + \frac{500\ m^3}{x}.$$

Check units!! $\underbrace{[x]}_{\text{"units of } x\text{"}} = m$, so $[A] = \left[\underbrace{2x^2}_{m^2} + \underbrace{\frac{500\ m^3}{x}}_{m^2}\right] = m^2$ ✓.

State the domain for A:

(3) $A = 2x^2 + \frac{500\ m^3}{x}$, $\quad 0 < x < +\infty$ or $x \in (0, +\infty)$ — either one

Check: If $x = 5\ m$, then by (1*), $h = \frac{125\ m^3}{5^2\ m^2} = 5\ m$ as well, so by (2),
$A = 2(5\ m)^2 + 4(5\ m)(5\ m) = 10\ m^2 + 100\ m^2 = 110\ m^2$
and by (3), $A = 2x^2 + \frac{500\ m^3}{x} = 2(5\ m)^2 + \frac{500\ m^3}{5\ m} = 10\ m^2 + 100\ m^2 = 110\ m^2$ ✓.

1. Draw a picture
2. Write down what we know
3. ______ , ______ want to find, in terms of what we know.
4. Reality check! — units, domain, etc.

Compos'n.

e.g., $f(x) = \sqrt{x}$, $g(x) = 1 - x^2$.

Then $f(g(x)) = \sqrt{1-x^2}$, but be careful — $1-x^2$ must be ≥ 0,

so want $x^2 \leq 1 \Rightarrow x \in [-1, 1]$.

Also, $\underbrace{g(f(x))}_{= h(x)} = 1 - (\sqrt{x})^2 = 1 - x$, defined for $x \geq 0$ (because f was only def. there)

Modelling — next pg.

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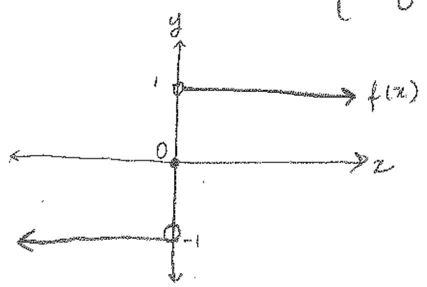
5 MIN. BREAK

see next pg.

1.17 - Find range of $f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$



$D = \mathbb{R}$

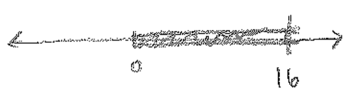
$R = \{-1, 0, 1\}$.

1.33 - Find largest possible domain, $f(x) = \sqrt{4 - \sqrt{x}}$

1) $x \geq 0$

2) $4 - \sqrt{x} \geq 0 \Rightarrow \sqrt{x} \leq 16 \Rightarrow x \leq 16$

$D = [0, 16]$.



"domain"

"subset"

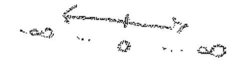
"real # line"

Def. A real-valued function f defined on a set $D \subseteq \mathbb{R}$

is a rule that assigns t/each input $x \in D$

"is an elt. of"

"in"



$\boxed{\text{exactly one}}$ output real number

$f(x) \in \mathbb{R}$.

range.

output $\longrightarrow$ dependent vars.

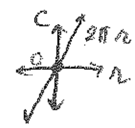
input $\longrightarrow$ indep. variables

- $f(x) = \sqrt{x}$ $\quad D = [0, +\infty)$

[— closed endpoint – i.e., interval includes the boundary

) — open endpoint – interval does not include this bdry. (N.B. – $\pm\infty$ always open endpoints)

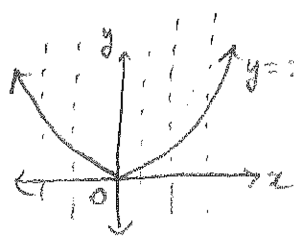
e.g., 

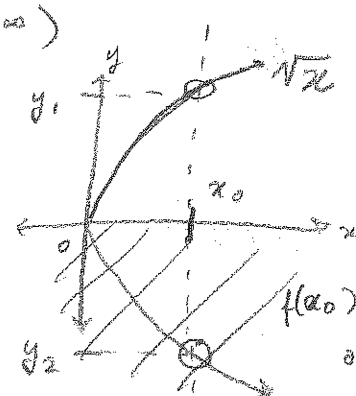
$C(r) = 2\pi r$. 

$D = [0, +\infty)$

$R = [0, +\infty)$

- ~~$f(7) = 5$ and $f(7) = 12$~~





$f(x_0) = y_1$

or ?

$f(x_0) = y_2$

$R = \{ x \in \mathbb{R} : x \geq 0 \}$

$\{ \quad : \quad \}$

Better?

$f(x) = \cos(x)$