Lecture 10: June 25, 2014.

TODAY: Higher derivatives, concavity 4.6
Curve sketching, asymptotes 4.7

Higher derivatives.

- The SECOND DERIVATIVE of f is the derivative of $f'(x)$, and is denoted $f''(x)$, or by $\frac{d^2y}{dx^2}$ (note the positions of the "2".)
- The THIRD DERIVATIVE of f is the derivative of f'', and is denoted $f'''(x)$, or $\frac{d^3y}{dx^3}$.
- The n^{th} DERIVATIVE of f is the derivative of the $n-1^{st}$ derivative, and is denoted $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$.

Example 1, p.266

Find the 1^{st} 4 derivs of $f(x) = 2x^3 + \frac{1}{x^2} + 16x^{7/2}$.

$$f'(x) = \frac{d}{dx}[f(x)] = 6x^2 \frac{-2}{x^3} + 56x^{5/2}$$

$$f''(x) = \frac{d}{dx}[f'(x)] = 12x + \frac{6}{x^4} + 140x^{3/2}$$

$$f'''(x) = \frac{d}{dx}[f''(x)] = 12 - \frac{24}{x^5} + 210\sqrt{x}$$

$$f^{IV}(x) = \frac{d}{dx}[f'''(x)] = +\frac{120}{x^6} + \frac{105}{\sqrt{x}}.$$

$\rightarrow$ or $f^{(4)}(x)$

For an implicit function:

Example 2, p. 266

$$\boxed{x^2 - xy + y^2 = 9}$$

$$\frac{d}{dx}\left[x^2 - xy + y^2\right] = \frac{d}{dx}[9] \qquad \text{RHS} = 0$$

$$\text{LHS} = 2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x - y + (2y - x)\frac{dy}{dx}$$

So, $2x - y + (2y-x)\frac{dy}{dx} = 0 \Rightarrow \boxed{\frac{dy}{dx} = \frac{2x-y}{2y-x}}$. 1^{st} deriv.

To find $\frac{d^2y}{dx^2}$, differentiate again:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{2x-y}{2y-x}\right] = \frac{\frac{d}{dx}[2x-y](2y-x) - \frac{d}{dx}[2y-x](2x-y)}{(2y-x)^2}$$

$$= \frac{\left(2 - \frac{dy}{dx}\right)(2y-x) - \left(2\frac{dy}{dx} - 1\right)(2x-y)}{(2y-x)^2}$$

$$= \frac{4y - \cancel{2x} - \cancel{2y\frac{dy}{dx}} + x\frac{dy}{dx} + 4x\frac{dy}{dx} + \cancel{2y\frac{dy}{dx}} + \cancel{2x} - y}{(2y-x)^2}$$

$$= \frac{3y + 3x\boxed{\frac{dy}{dx}}}{(2y-x)^2}$$

* NOTE THE SYMMETRY IN (x,y) *

$$= \frac{3y + 3x\frac{(2x-y)}{(2y-x)}}{(2y-x)^2} = \frac{3y(2y-x) + 3x(2x-y)}{(2y-x)^3} =$$

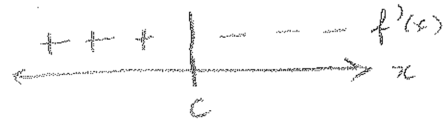
$$= -\frac{6y^2 - 3xy + 6x^2 - 3xy}{(2y-x)^3} = -\frac{6\boxed{(x^2 - xy + y^2)}}{(2y-x)^3} = \boxed{\frac{-54}{(2y-x)^3}}.$$

A note on maxima & minima ——

SUPPOSE $x = c$ IS A CRITICAL POINT OF f WHERE $f'(c) = 0$.

If $f''(c) < 0$, then $f'(x)$ is decreasing at $x = c$ — so,

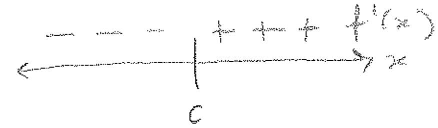
it must look like



A local maximum.

If $f''(c) > 0$, then $f'(x)$ is increasing at $x = c$ — so,

it must look like



A local minimum.

SECOND DERIVATIVE TEST.

Suppose $f'(c) = 0$, and $f''(x)$ exists at $x = c$.

IF $f''(c) < 0$, THEN $f(c)$ IS A LOCAL MAXIMUM OF f.

IF $f''(c) > 0$, THEN $f(c)$ IS A LOCAL MINIMUM OF f.

IF $f''(c) = 0$, THEN INCONCLUSIVE — USE OTHER TEST.

Example 3
p. 268

$f(x) = x^3 - 3x^2 + 3 \Rightarrow f'(x) = 3x^2 - 6x = 3x(x-2)$.

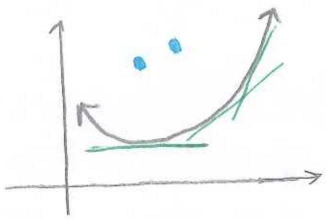
$f''(x) = 6x - 6 = 6(x-1)$

So critical points are at $x = 0$, $x = 2$.

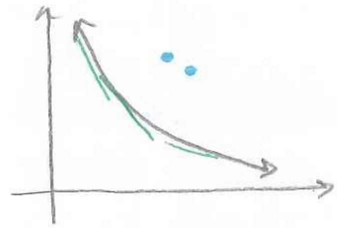
Now, $f''(0) = 6(0-1) < 0$ and $f''(2) = 6(2-1) > 0$, so $f(0) = 3$ is a local max and $f(2) = -1$ is a local minimum.

Concavity.

- "smiley" curves are concave up:



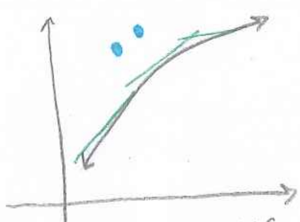
INCREASING



DECREASING

These curves lie above the tangent line

- "frowney" curves are concave DOWN:



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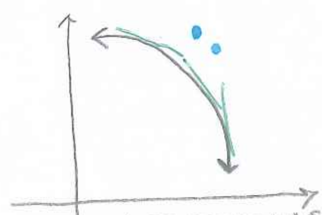


DECREASING

These curves lie below the tangent line.

NOTE:

$f(x)$ IS CONCAVE UP WHEN $f''(x) > 0$
$f(x)$ IS CONCAVE DOWN WHEN $f''(x) < 0$

This could be (is) really useful in curve sketching!

Lec 10, ct'd.

Points where f CHANGES CONCAVITY are called POINTS OF INFLECTION, and are found similarly to how extrema are found:

1. Find $f'(x)$, $f''(x)$
2. Find the "possible points of inflection" (or "P.P.I.") by finding where $f''(x) = 0$ or is undefined.
3. Test the sign of f'' (SECOND deriv.) on the intervals to the left & right of the P.P.I.
4. $+++ \mid --- \quad f''(x)$, with c on the x-axis ($\cup\cup\cup$ to the left, $\frown\frown\frown$ to the right): is "concave up - to - concave down" INFLECTION PT.

   $--- \mid +++ \quad f''(x)$, with c on the x-axis ($\frown\frown\frown$ to the left, $\cup\cup\cup$ to the right): is "concave down - to - concave up", INFLECTION PT.

SOMETIMES, c isn't either - it's not an inflec'n pt.

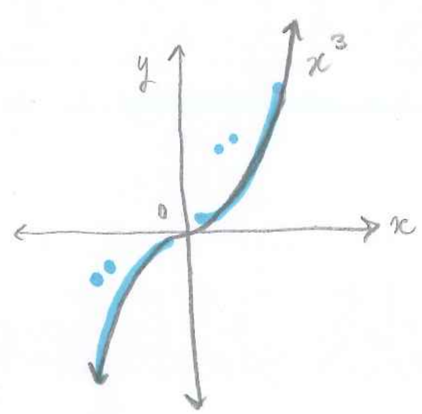
Example. $f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$.

C.P.: $x = 0$

P.P.I: $x = 0$.

$+++ \quad +++ \quad f'(x)$ on the x-axis (-1, 0, 1): so C.P. is not an extremum

$--- \quad +++ \quad f''(x)$ on the x-axis (-1, 0, 1): so P.P.I. IS an inflec'n pt, and concavity changes from down-to-up.



Example 6, p. 273

Sketch the graph of $f(x) = 8x^5 - 5x^4 - 20x^3$, indicating local extrema, inflection points, and concavity.

$f'(x) = 40x^4 - 20x^3 - 60x^2 = 20x^2(2x^2 - x - 3) = 20x^2(2x-3)(x+1)$

$f''(x) = 160x^3 - 60x^2 - 120x = 20x(8x^2 - 3x - 6)$.

Critical points: $f'(x) = 0$ at $x = 0$, $x = 3/2$, $x = -1$

$f'(x)$ always defined.

P.P.I: $f''(x) = 0$ at $x = 0$, $x = \dfrac{3 \pm \sqrt{3^2 - 4(8)(-6)}}{2(8)} = \dfrac{3 \pm \sqrt{201}}{16}$

$f''(x)$ always defined.

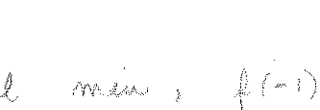
Use the 2nd derivative test on the c.p:

$f''(0) = 0 \Rightarrow$ test inconclusive $\Rightarrow$ need 1st der. test for $x = 0$.

$f''(3/2) = 20(\frac{3}{2})(8(\frac{3}{2})^2 - 3(\frac{3}{2}) - 6) = 30(18 - 6 - 4.5) > 0$

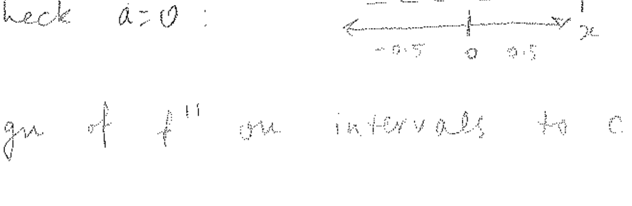
$f''(-1) = 20(-1)(8(-1)^2 - 3(-1) - 6) = -20(8+4-6) < 0$

Thus, $f(3/2)$ is a local min, $f(-1)$ a local max.

Check $a = 0$: 

so not an extremum.

Check sign of f'' on intervals to classify the P.P.I.



So, $x = 0$, $\dfrac{3 \pm \sqrt{201}}{16}$ are all I.P.

↓ ct'd.

Example 6, p. 273 ct'd.

So, local max at $(-1, f(-1)) = (-1, 7)$

local min at $(3/2, f(3/2)) = (3/2, -32.0625)$

Increasing on $(-\infty, -1) \cup (3/2, +\infty)$

Decreasing on $(-1, 3/2)$

Inflection points at $\left(\frac{3-\sqrt{201}}{16}, f\left(\frac{3-\sqrt{201}}{16}\right)\right) \approx (-0.7, 4.3)$

$\left(\frac{3+\sqrt{201}}{16}, f\left(\frac{3+\sqrt{201}}{16}\right)\right) \approx (1.07, -19.98)$

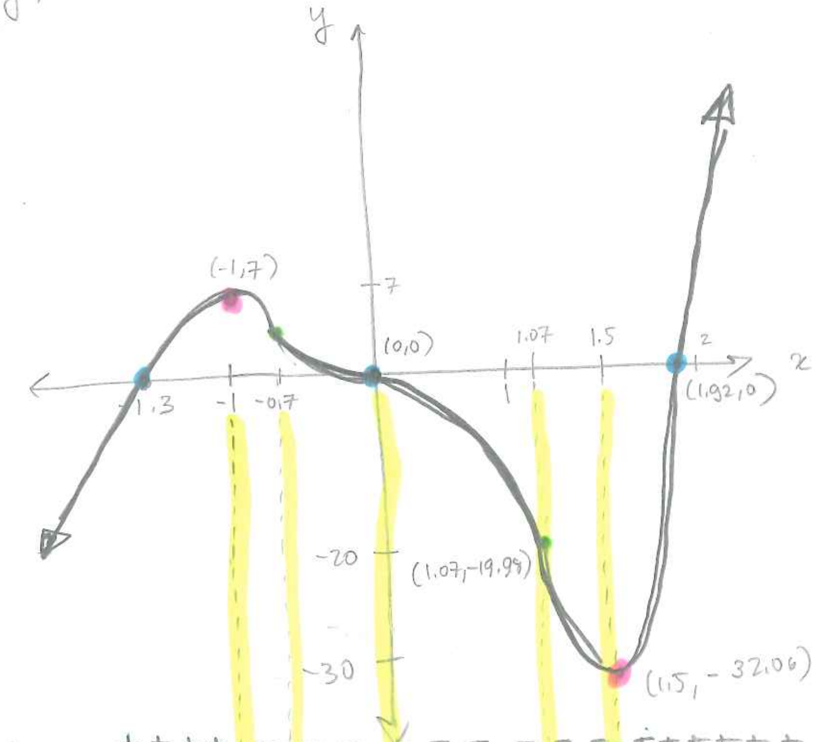
$(0, f(0)) = (0,0)$

Concave up on $(-0.7, 0) \cup (1.07, +\infty)$

down on $(-\infty, -0.7) \cup (0, 1.07)$

Recall last lec.

Finally, the intercepts: $(0,0)$, $(-1.3, 0)$, $(1.9, 0)$



INC/DEC: + + + + − − − − − − − − − − + + + + + +

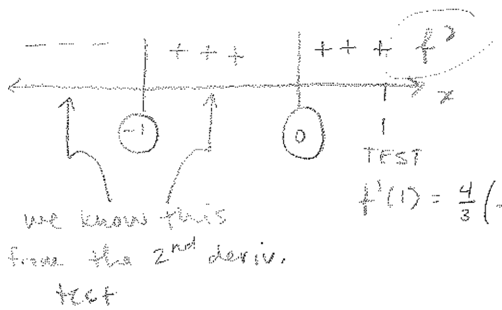
Example. 7, p.274 Sketch the graph of $f(x) = 4x^{1/3} + x^{4/3}$. Indicate local extrema, inflection points, and concave structure.

Well, $f'(x) = \frac{4}{3}x^{-2/3} + \frac{4}{3}x^{1/3} = \frac{4}{3}\left(\frac{1}{x^{2/3}} + x^{1/3}\right) = \frac{4}{3}\left(\frac{1+x}{x^{2/3}}\right)$, and $f''(x) = -\frac{8}{9}x^{-5/3} + \frac{4}{9}x^{-2/3} = \frac{4}{9}\left[\frac{1}{x^{2/3}} - \frac{2}{x^{5/3}}\right] = \frac{4}{9}\left[\frac{x-2}{x^{5/3}}\right]$.

So, <u>C.P.</u> : $x=0$, $x=-1$

<u>P.P.I.</u> : $x=0$, $x=2$

Second derivative test for c.p. is inconclusive for $x=0$, and $f''(-1)>0$, so $f(-1) = 4(-1)^{1/3} + (-1)^{4/3} = -3$ <u>is a local minimum</u>. Need to test the sign of f' on the intervals around $x=0$:



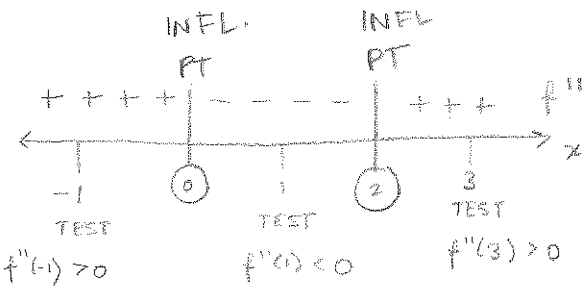
$x=0$ <u>not</u> an extremum!

$f(-1) = 4(-1)^{1/3} + (-1)^{4/3} = -4+1 = -3$

<u>Inc/Dec</u>

f is thus increasing on $(-1,+\infty)$ and decreasing on $(-\infty,-1)$.

Now, test the sign of f'' around the P.P.I. to obtain the concavity:



$f(2) = 4(2)^{1/3} + 2^{4/3} = 6\sqrt[3]{2}$

$f(0)=0$

<u>Concavity</u> f is concave up on $(-\infty,0)\cup(2,+\infty)$ and concave down on $(0,2)$.

Note that $\lim_{x\to-\infty} f(x) = +\infty = \lim_{x\to+\infty} f(x)$ (the "limiting behavior").

<u>Intercepts</u> : $(0,0)$ is the y-intercept.

$0 = 4x^{1/3} + x^{4/3} \Rightarrow \underbrace{-4x^{1/3} = x^{4/3}}_{x=0 \;\Rightarrow\; (0,0) \text{ is an } x\text{-intercept}} \Rightarrow x=-4$. So $(-4,0)$ is an x-intercept.

Lec 10, ct'd.

So, what we know:

f increasing on $(-1, 0)$, decreasing on $(-\infty, -1) \cup (0, +\infty)$

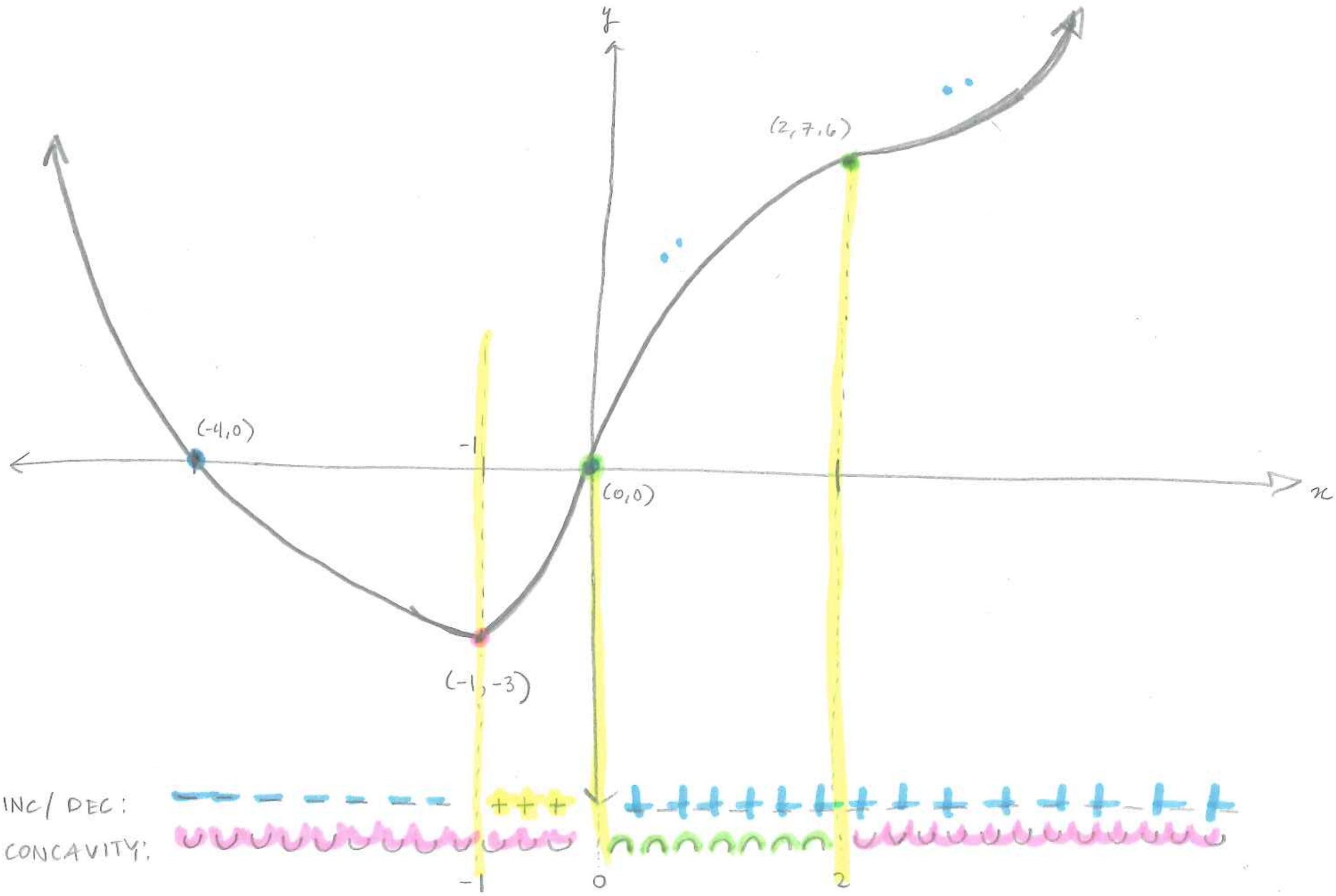
f concave up on $(-\infty, 0) \cup (2, +\infty)$, concave down on $(0, 2)$

f has a local minimum at $(-1, f(-1)) = (-1, -3)$

f has inflection points at $(0,0)$ and $(2, 6\sqrt[3]{2}) \approx (2, 7.6)$

f has y-intercept $(0,0)$ and x-intercepts $(0,0)$, $(-4,0)$.

So, the graph:



Lec 10, ct'd.

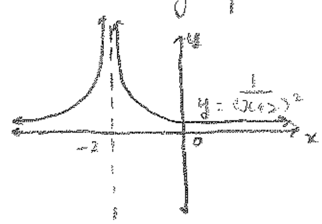
Another curve-sketching technique: ASYMPTOTES.

Def. The line $x=a$ is a VERTICAL ASYMPTOTE of $f(x)$ provided either:

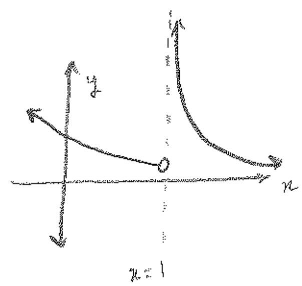
$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty \quad \text{or both.}$$

e.g., $f(x)=\dfrac{1}{(x+2)^2}$ has a vertical asymptote at $x=-2$, because

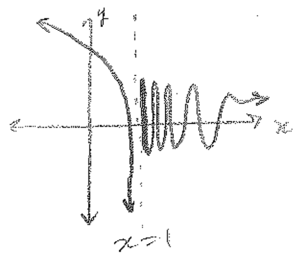
$$\lim_{x\to -2} = +\infty \quad ;$$



e.g., a "right-hand only" asymptote:



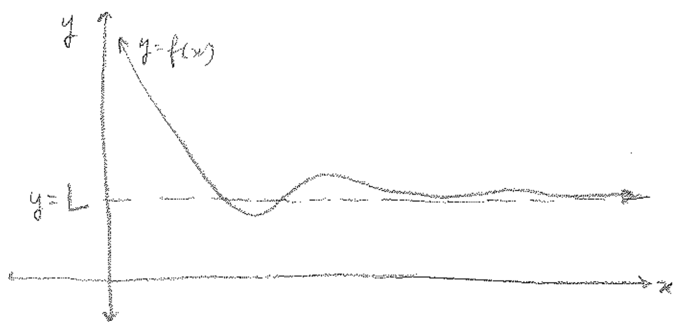
a "left-hand only":



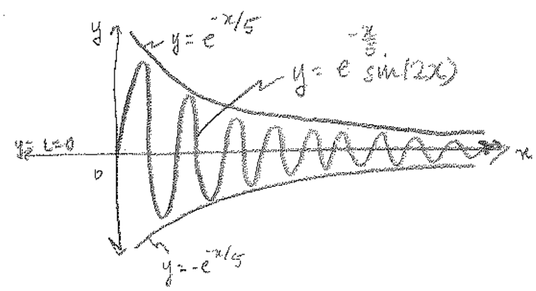
Def. The line $y=L$ is a HORIZONTAL ASYMPTOTE of $f(x)$ if either

$$\lim_{x\to +\infty} f(x) = L \quad \text{or} \quad \lim_{x\to -\infty} f(x) = L \quad \text{or both.}$$

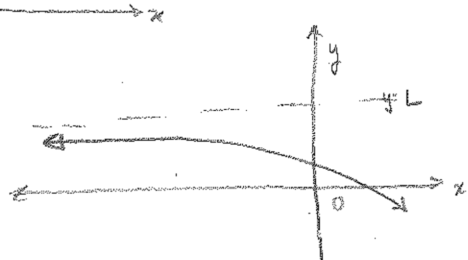
e.g.,



e.g.,



e.g.,



Def. The nonvertical line $y = mx + b$ is a SLANT ASYMPTOTE of $f(x)$ if either:

$$\lim_{x \to +\infty} \left[f(x) - (mx+b) \right] = 0 \quad \text{or} \quad \lim_{x \to -\infty} \left[f(x) - (mx+b) \right] = 0 \quad \text{or both.}$$

e.g., $f(x) = \dfrac{x^2 + x - 1}{x - 1}$. Do out the long division:

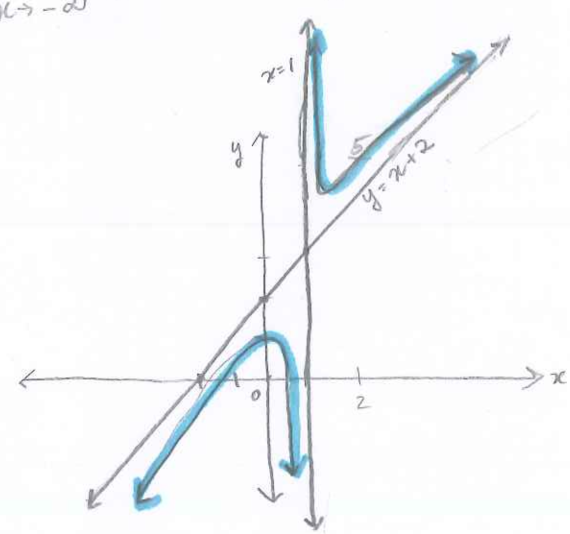
$$\begin{array}{r} x + 2 \\ x-1 \,\overline{)\, x^2 + x - 1} \\ \underline{-x^2 + x \quad\;\;} \\ 2x - 1 \\ \underline{-2x + 2} \\ 1 \end{array} \leftarrow \text{REMAINDER}$$

$$\Rightarrow \quad \frac{x^2 + x - 1}{x - 1} = \underbrace{x + 2}_{\text{QUOTIENT}} + \frac{\overset{\text{REM}}{1}}{x - 1}.$$

Thus, $f(x) = x + 2 + \dfrac{1}{x-1}$, so let $y = x + 2$ — then

$$\lim_{x \to +\infty} \left[f(x) - (x+2) \right] = \lim_{x \to +\infty} \left[\frac{1}{x-1} \right] = 0,$$ so $y = x + 2$ is a slant asymptote

and $$\lim_{x \to -\infty} \left[f(x) - (x+2) \right] = \lim_{x \to -\infty} \left[\frac{1}{x-1} \right] = 0,$$ so actually, both conditions hold:



ALSO, $x = 1$ is a vertical asymptote

So, we have a pretty good curve sketching strategy (p. 285 of text):

1. Find $f'(x)$ and use it to get the critical points – where $f'(x) = 0$ or $f'(x)$ doesn't exist. Note whether the tan. line is horizontal, vertical, or nonexistent at each c.p.

2. Determine the intervals where f increases / decreases – also tells you local extrema.

3. Find $f''(x)$, use it to get $\underbrace{\text{possible points of inflection}}^{\text{"PPI"}}$ – where either $f''(x) = 0$ or $f''(x)$ doesn't exist.

4. Determine the intervals on which f is concave up / down – also tells you which of the P.P.I. are actually P.I.

5. Find the y-intercept by setting $x = 0$ and the x-intercept by setting $y = 0$.

6. Plot and label the c.p., P.P.I., and intercepts. If it helps, label the intervals on which f is increasing / decreasing, and where f is concave up / down.

7. Determine the asymptotes (if any), the discontinuities (if any), and (* IMPORTANT *) the behavior of f and of f' near discontinuities of f.

8. Determine the limiting behavior $\lim_{x \to \pm\infty} f(x)$.

9. Join the plotted points w/ a curve that is consistent with the information in 1–8.

THESE STEPS ... ANY ORDER, OMITTING UNNECESSARY STEPS

Example 8, p. 285

Sketch the graph of $f(x) = \dfrac{2+x-x^2}{(x-1)^2}$.

Notice, $f(x) = \dfrac{-(x-2)(x+1)}{(x-1)^2}$ has $\lim_{x \to 1} f(x) = +\infty$, so $x=1$ a VERTICAL asymptote.

Also, $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} f(x)\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to \pm\infty} \dfrac{\overset{0}{\frac{2}{x^2}} + \overset{0}{\frac{1}{x}} - 1}{(1-\underset{0}{\frac{1}{x}})^2} = -1$, so $y=-1$ is a HORIZONTAL asymptote, on both sides.

Also, $f(0) = \dfrac{(0-2)(0+1)}{(0-1)^2} = \dfrac{-(-2)(1)}{(1)} = 2$, and $0 = -(x-2)(x+1)$ when $x=-1$ or $x=2$,

So the y-intercept is $(0,2)$ and the x-intercepts are $(-1,0)$ and $(2,0)$.

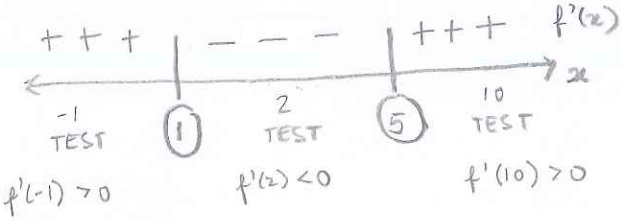
This took care of the asymptotes, limiting behavior, behavior of f near the discontinuity, and the x- and y-intercepts.

Now, $f'(x) = \dfrac{\frac{d}{dx}[2+x-x^2](x-1)^2 - \frac{d}{dx}[(x-1)^2](2+x-x^2)}{(x-1)^4} = \dfrac{(1-2x)(x-1)^2 - 2(x-1)(2+x-x^2)}{(x-1)^4} =$

$= \dfrac{(1-2x)(x-1) - 2(2+x-x^2)}{(x-1)^3} = \dfrac{x - \cancel{2x^2} - 1 + \cancel{2x} - 4 - \cancel{2x}\, \cancel{2x^2}}{(x-1)^3} = \dfrac{x-5}{(x-1)^3}$,

which is undefined at $x=1$ and 0 at $x=5$. Well, $f(x)$ is also undefined at $x=1$, so there is no tangent line there, and the tangent line is horizontal at $x=5$.

Let's find the increasing / decreasing behavior:



$f(5) = \dfrac{2+5-25}{16} = -\dfrac{9}{8}$ is a local min.

And $\lim_{x \to 1^-} f'(x) = -\infty$

$\lim_{x \to 1^+} f'(x) = +\infty$

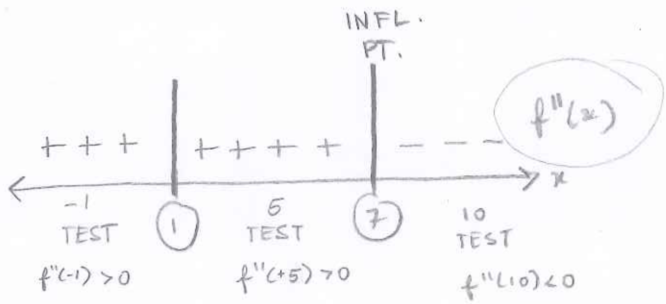
f inc. on $(-\infty, 1) \cup (5, +\infty)$

$\boxed{\text{Example 8, p.285, ct'd.}}$ So, we found the critical points, the local min, and the intervals on which f is inc/dec.

Now, $f''(x) = \dfrac{\frac{d}{dx}[x-5](x-1)^3 - \frac{d}{dx}[(x-1)^3](x-5)}{(x-1)^6} = \dfrac{(x-1)^{\cancel{3}} - 3\cancel{(x-1)^2}(x-5)}{(x-1)^{\cancel{6}4}} = \dfrac{(x-1) - 3(x-5)}{(x-1)^4} =$

$= \dfrac{-2x+14}{(x-1)^4} = \dfrac{-2(x-7)}{(x-1)^4}$, which is undefined at $x=1$ and

is zero at $x=7$ — so the P.P.I. are $x=1$ and $x=7$.



$f(7) = \dfrac{2+7-49}{36} = \dfrac{-10}{9}$ is a point of inflec'n

So, f concave up on $(-\infty, 7)$ and conc. down on $(7, +\infty)$

- $x=1$ is a vertical asymptote — both $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x) = +\infty$
- $y=-1$ is a horizontal asymptote — both $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = -1$
- y-intercept: $(0,2)$, x-intercepts: $(-1,0)$, $(2,0)$
- $(5, -9/8)$ is a local min. (horiz. tan. line)
- there is no tan. line at $x=1$
- f increasing on $(-\infty, 1) \cup (5, +\infty)$, dec. on $(1, 5)$
- $(7, -10/9)$ is an inflec'n pt. of f
- f concave up on $(-\infty, 7)$ and conc. down on $(7, +\infty)$.

$$f(x) = \frac{2+x-x^2}{(x-1)^2}$$