TODAY: 4.8, 4.9 } Indeterminate Forms & L'Hôpital's rule.

ANNOUNCEMENTS:
- Final Exam WEDNESDAY, JULY 02 from 6-8 p.m. here, in OH 109.
- Extra office hours: TUESDAY (1 JULY) (1-3 p.m.), OH 109.
- No homework or WebWork this week.
- Remember the "deal" with the final exam — if you do better on the final than you did on the midterm, then the final replaces the midterm.
- Review for final in 2^{nd} half today.

4.8: Indeterminate forms & l'Hôpital's Rule.

Def. An INDETERMINATE FORM is a certain type of expression with a limit that is not evident by inspection, e.g.:

- "$\frac{0}{0}$" form: $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} \overset{?}{=} \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} x}$ "$\frac{0}{0}$"

- "$\frac{\infty}{\infty}$" form: $\displaystyle\lim_{x\to 0^+} \frac{\ln(2x)}{\ln(x)} \overset{?}{=} \frac{\lim_{x\to 0^+} \ln(2x)}{\lim_{x\to 0^+} \ln(x)}$ "$\frac{-\infty}{-\infty}$"

→ Quotation marks important — lets us know we're not dealing with numbers, but forms.

IMPORTANT EXAMPLE $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ has form "1^∞"

$$= \exp\left\{ \ln\left[\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \right] \right\}$$

$$= \exp\left\{ \lim_{x\to\infty} \ln\left[\left(1+\frac{1}{x}\right)^x \right] \right\}$$

$$= \exp\left\{ \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right) \right\}$$ limit has form "$\infty \cdot 0$"

$$= \exp\left\{ \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} \right\}$$ limit has form "$\frac{0}{0}$" — apply l'Hôp:

$$= \exp\left\{ \lim_{x\to\infty} \frac{\frac{d}{dx}\left[\ln\left(1+\frac{1}{x}\right)\right]}{\frac{d}{dx}\left[1/x\right]} \right\}$$

$$= \exp\left\{ \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right)}{(-1/x^2)} \right\}$$

$$= \exp\left\{ \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} \right\}$$

$$= \exp\{1\}$$

$$= e.$$

So, $$\boxed{e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x}$$

- " 0^0 ", " ∞^0 ", " 1^∞ "

e.g., $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$ has form " 1^∞ ".

$e^{\ln(\text{anything})} = \text{anything}$

$$= \exp\left[\ln\left[\lim_{x\to 0} (\cos x)^{1/x^2}\right]\right]$$

$$= \exp\left[\lim_{x\to 0} \ln\left[(\cos x)^{1/x^2}\right]\right]$$

$\ln(a^b) = b\ln(a)$

$$= \exp\left[\lim_{x\to 0} \frac{\ln[\cos x]}{x^2}\right]$$

① limit has form " $\frac{0}{0}$ "

② $f(x) = \ln[\cos(x)]$
$f'(x) = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$

③ $g(x) = x^2$

④ $g'(x) = 2x \neq 0$ in a deleted nbhd of $x=0$

$$= \exp\left[\lim_{x\to 0} \frac{\frac{d}{dx}[\ln(\cos x)]}{\frac{d}{dx}[x^2]}\right]$$

$$= \exp\left[\lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{2x}\right]$$

$$= \exp\left[\lim_{x\to 0} -\frac{\sin x}{2x\cos x}\right]$$

① limit has form " $\frac{0}{0}$ "

② $f(x) = -\sin(x)$ diff'ble in a (deleted) nbd. of $x=0$

③ $g(x) = 2x\cos x$ — " —

④ $g'(x) = 2\cos x - 2x\sin x \neq 0$ in a (deleted) nbd. of $x=0$ ✓

$$= \exp\left[\lim_{x\to 0} \frac{\frac{d}{dx}[-\sin x]}{\frac{d}{dx}[2x\cos x]}\right]$$

$$= \exp\left[\lim_{x\to 0} \frac{-\cos x}{-2x\underbrace{\sin x}_{0} + 2\cos x}\right]$$

$$= \exp\left[-\frac{1}{2}\right] = \boxed{\frac{1}{\sqrt{e}} = \lim_{x\to 0} (\cos x)^{1/x^2}}$$

4.9: Ind. forms, ct'd.

7

- "$\infty - \infty$"

$$\underbrace{\lim_{x\to 0}\frac{1}{x}}_{\infty} - \underbrace{\lim_{x\to 0}\frac{1}{\sin x}}_{\infty} \quad \neq$$

e.g., $\underbrace{\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)}_{\text{has form "}\infty-\infty\text{"}} = \lim_{x\to 0}\frac{\sin x - x}{x\sin x}$ has form "$\frac{0}{0}$" now!

① has form "$\frac{0}{0}$", can apply l'Hôpital

$$= \lim_{x\to 0}\frac{\frac{d}{dx}[\sin x - x]}{\frac{d}{dx}[x\sin x]}$$

② $f(x) = \sin x - x$ is diff'ble in all nbhds. of $a=0$

③ $g(x) = x\sin x$ is diff'ble in all nbds of $a=0$

$g'(x) = \sin x + x\cos x$

$$= \lim_{x\to 0}\frac{\cos x - 1}{\sin x + x\cos x}$$

① still "$\frac{0}{0}$"

④ $g'(x) = \sin x + x\cos x \neq 0$ in some nbd of $a=0$.

In particular, on $(-\frac{1}{2},\frac{1}{2})\setminus\{0\}$, $g'(x)\neq 0$.

set minus

$$= \lim_{x\to 0}\frac{\frac{d}{dx}[\cos x - 1]}{\frac{d}{dx}[\sin x + x\cos x]}$$

$$= \lim_{x\to 0}\frac{-\sin x}{\cos x + \cos x - x\sin x}$$

② $f(x) = \cos x - 1$ is diff'ble

③ $g(x) = \sin x + x\cos x$ is diff'ble

④ $g'(x) = 2\cos x - \sin x \neq 0$ on some deleted nbd. of $x=0$.

$$\stackrel{?}{=} \frac{\lim_{x\to 0} -\sin x}{\lim_{x\to 0} \cos x + \cos x - x\sin x}$$

$$= \frac{0}{2} = 0.$$

$$\boxed{\lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{\sin x}\right] = 0}$$

- How to evaluate limits like these?

l'Hospital ←

THEOREM. ("L'Hôpital's Rule" — Bernoulli, 1696)

If f and g are differentiable in some nbd. (neighborhood) of the point $x = a$, AND $g'(x) \neq 0$ in some nbd. of $x = a$ (except possibly at a itself), AND if $\lim_{x \to a} \frac{f(x)}{g(x)}$ has the indeterminate form "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$",

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided that the limit on the right exists or is $\pm\infty$.

NOTES:

- Works when $a = \pm\infty$ (i.e., $\lim_{x \to \infty} \frac{f(x)}{g(x)}$)

⟼ • $\lim_{x \to a} \frac{f(x)}{g(x)}$ MUST be an indeterminate form !!

(will see an example showing why)

⟼ • $\lim_{x \to a} \frac{f'(x)}{g'(x)} \neq \lim_{x \to a} \left(\frac{f(x)}{g(x)}\right)'$ (i.e., DON'T USE QUOTIENT RULE — this is not about the derivative of the quotient here — it really is the quotient of derivs)

4.9: Other indeterminate forms

- "$0 \cdot \infty$"

e.g., $\lim_{x\to\infty} x \ln\left(\frac{x-1}{x+1}\right) = \underbrace{\left(\lim_{x\to\infty} x\right)\left(\ln\left(\lim_{x\to\infty} \frac{x-1}{x+1}\right)\right)}_{\infty \cdot 0}$

$$\frac{x-1}{x+1} = \frac{1-1/x}{1+1/x} \to 1$$

Check: $\lim_{x\to\infty} x = \infty$

$$\lim_{x\to\infty} \ln\left(\frac{x-1}{x+1}\right) = \ln\left(\lim_{x\to\infty} \frac{x+1}{x-1}\right)$$

$$= \ln\left(\lim_{x\to\infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right)$$

$$= \ln(1)$$

$$= 0.$$

Rewrite: $\lim_{x\to\infty} \overset{\frac{1}{(1/x)}}{x} \ln\left(\frac{x-1}{x+1}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x-1}{x+1}\right)}{1/x}$, and this is in a form where we can use l'Hôpital:

Check: ① $\lim_{x\to\infty} \ln\left(\frac{x-1}{x+1}\right) = 0 = \lim_{x\to\infty} \frac{1}{x}$, so form "$\frac{0}{0}$"

② $f(x) = \ln\left(\frac{x-1}{x+1}\right)$ diff'ble 'around' $+\infty$ $\quad f'(x) = \frac{x+1}{x-1}\frac{d}{dx}\left(\frac{x-1}{x+1}\right)$

$$= \frac{x+1}{x-1}\,\frac{x+1-x+1}{(x+1)^2}$$

$$= \frac{2}{(x-1)(x+1)}$$

③ $g(x) = \frac{1}{x}$ diff'ble

④ $g'(x) = \frac{-1}{x^2} \neq 0$ for, e.g., $x \in (10, +\infty)$

So, by l'Hôpital's Rule:

$$\lim_{x\to\infty} x\ln\left(\frac{x-1}{x+1}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x-1}{x+1}\right)}{1/x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left[\ln\left(\frac{x-1}{x+1}\right)\right]}{\frac{d}{dx}\left[1/x\right]} =$$

$$= \lim_{x\to\infty} \frac{\frac{x+1}{x-1}\left(\frac{x+1-x+1}{(x+1)^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{-2x^2}{(x+1)(x-1)} = \lim_{x\to\infty} \frac{-2x^2 \cdot \frac{1}{x^2}}{(x^2-1)\cdot\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{-2}{1-\cancel{1/x^2}_{\to 0}} = -2.$$

$\boxed{\text{Example } \\ 5, \text{p.} 297}$

$$\lim_{x\to\infty} \frac{x^2}{e^x}.$$

Check: ① $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} e^x = \infty$, so IND. FORM "$\frac{\infty}{\infty}$"

② $f(x) = x^2$ is diff'ble everywhere ("around $+\infty$")

③ $g(x) = e^x$ is diff'ble —— " ——

④ $g'(x) = e^x$ is nonzero —— " ——

so, $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^x]}$

$$= \lim_{x\to\infty} \frac{2x}{e^x} \overset{?}{=} \frac{\lim_{x\to\infty} 2x}{\lim_{x\to\infty} e^x}$$

Check: ① $\lim_{x\to\infty} 2x = \infty$, $\lim_{x\to\infty} e^x = \infty$, so IND. FORM "$\frac{\infty}{\infty}$"

② $f(x) = 2x$ is diff'ble everywhere

③ $g(x) = e^x$ —— " ——

④ $g'(x) = e^x \neq 0$ everywhere

so, $\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[2x]}{\frac{d}{dx}[e^x]}$

$$= \lim_{x\to\infty} \frac{2}{e^x} \overset{\checkmark}{=} \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty} e^x} \longrightarrow 0.$$

Thus, $\lim_{x\to\infty} \frac{x^2}{e^x} = 0.$

Example 3, p. 295 $\displaystyle\lim_{x\to 0} \frac{\sin x}{x+x^2}$.

Check:

- $\displaystyle\lim_{x\to 0} \sin x = 0 = \lim_{x\to 0} x+x^2$

  So, YES, an indeterminate form "$\frac{0}{0}$"

- $\sin(x)$ diff'ble around $x=0$

- $x+x^2$ ——— " ———, and $\frac{d}{dx}[x+x^2] = 2x+1$ is nonzero in a nbd. of $x=0$.

So, apply l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x+x^2} = \lim_{x\to 0} \frac{\frac{d}{dx}[\sin x]}{\frac{d}{dx}[x+x^2]} = \lim_{x\to 0} \frac{\cos x}{1+2x} =$$

$$= \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} 1+2x}$$

$$= \frac{1}{1}$$

$$\boxed{= 1.}$$

$$= \lim_{x\to 0} \frac{\frac{d}{dx}[\cos x]}{\frac{d}{dx}[1+2x]}$$

$$= \lim_{x\to 0} \frac{-\sin(x)}{2}$$

$$= -\frac{1}{2}\lim_{x\to 0} \sin(x)$$

$$= 0.$$

WRONG

Example 1, p.294

$$\lim_{x\to 0} \frac{e^x - 1}{\sin(2x)}.$$

Note, $\lim_{x\to 0} e^x - 1 = 0 = \lim_{x\to 0} \sin(2x)$. Indeterminate form "$\frac{0}{0}$".

- $f(x) = e^x - 1$ is diff'ble around $x=0$
- $g(x) = \sin(2x)$ is diff'ble around $x=0$
- $g'(x) = 2\cos(2x) \neq 0$ around $x=0$.

So, l'Hôpital's rule applies, and

$$\lim_{x\to 0} \frac{e^x - 1}{\sin(2x)} = \lim_{x\to 0} \frac{\frac{d}{dx}\left[e^x - 1\right]}{\frac{d}{dx}\left[\sin(2x)\right]}$$

$$= \lim_{x\to 0} \frac{e^x}{2\cos(2x)}$$

$$\stackrel{\checkmark}{=} \frac{\lim_{x\to 0} e^x}{\lim_{x\to 0} 2\cos(2x)}$$

$$= \boxed{\frac{1}{2}}.$$