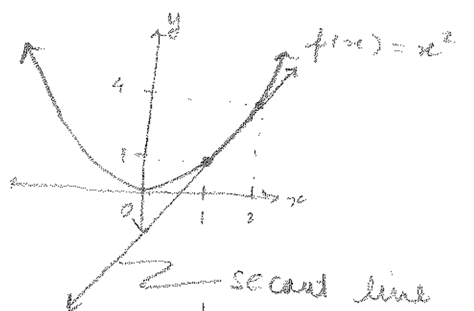
Week 2: Limits.

TURN ON MICROPHONE.

Secant line of a curve is a line that locally intersects two points on the curve. (Latin secare – "to cut")

e.g.,



secant line to $f(x)=x^2$, connecting $(1,1)$ to $(2,4)$

Suppose we'd just like to compute the slope of the secant line (we will see why one would want to do this later today).

For a general function $f(x)$, two points on the graph $y=f(x)$ are $(x, f(x))$ and $(x+\Delta x, f(x+\Delta x))$,

This is – for now – a fixed numerical value and never depends on x.

RECALL: capital Greek letter delta signifies a change in a quantity. Here, we talk abt. the change in x.

and the slope of the line connecting these two points is

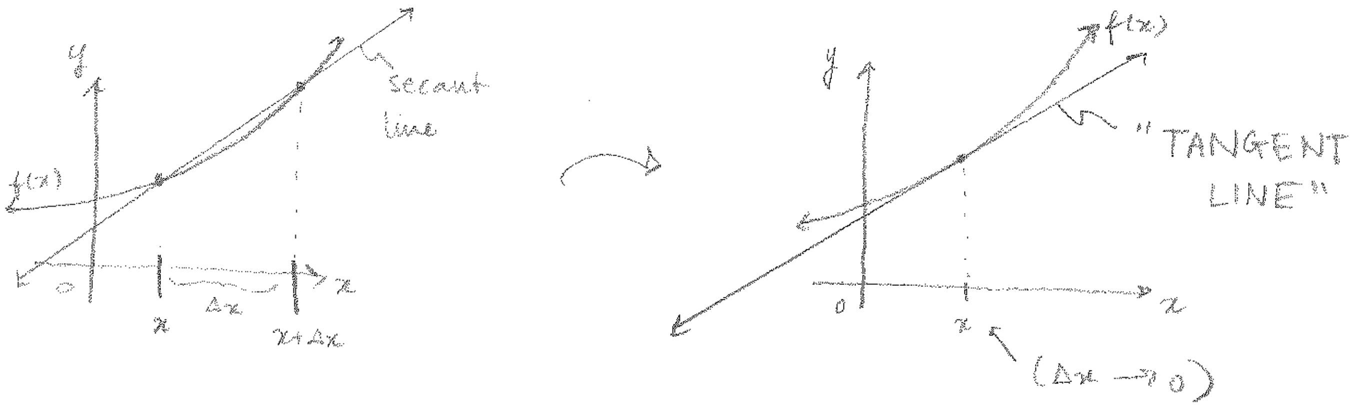
$$m_s = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{(\cancel{x}+\Delta x)-\cancel{x}} = \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

(yes, that's the same Δx as above – see why?)

See: Since Δx is fixed, and we ostensibly know f, then m depends only on x. Write $m_s(x) = \frac{f(x+\Delta x)-f(x)}{\Delta x}$ as ...

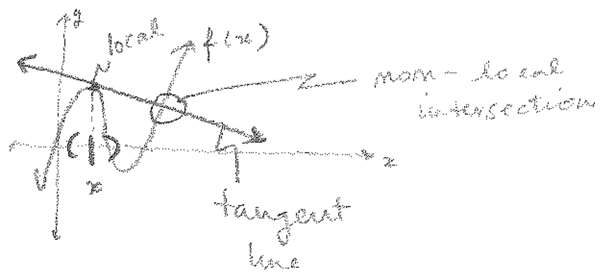
as its own function.

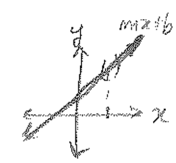
What happens when we let Δx shrink (to zero)?



A line tangent to f at the point x intersects the graph of f locally only at $(x, f(x))$. (Latin tangere — "to touch")

Q. Why "locally"? e.g.,





The slope of the tangent line is the limiting value of the slope of the secant line, as $\underbrace{\Delta x \to 0}_{\text{"approaches"}}$.

That is, $\underbrace{m(x)}_{\text{"Slope predictor fn."}} = \lim\limits_{\Delta x \to 0} m_s(x) = \boxed{\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}}$.

HOW DO WE CALCULATE LIMITS LIKE THIS?

(See: substitution gets us nowhere — end up w/ "$\frac{0}{0}$", which makes no sense mathematically.)

This was a general case — let's look at some examples.

e.g., $f(x) = x^2 + 7$.

$$m(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 + 7] - [x^2 + 7]}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{[\cancel{x^2} + 2x\Delta x + (\Delta x)^2 \cancel{+7}] - [\cancel{x^2} \cancel{+7}]}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\cancel{\Delta x} + (\Delta x)^{\cancel{2}}}{\cancel{\Delta x}}$$

$$= \underbrace{\lim_{\Delta x \to 0} [2x + \Delta x]} = \boxed{2x.}$$

See: substitution works here in evaluating the limit, because there are no "complications". This particular expression is fairly intuitive.

N.B. — We could have had any constant instead of 7,
("nota bene" - It.)
and it wouldn't have affected our answer, i.e., the slope predictor fn. for $f(x) = x^2 + c$ is $m(x) = 2x$.

e.g., $f(x) = 3x^2$.

$$m(x) = \boxed{\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} = \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 - 3x^2}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{3x^2} + 3(2x\,\Delta x) + 3(\Delta x)^2 \cancel{- 3x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(2x\cancel{\Delta x}) + 3(\Delta x)^{\cancel{2}}}{\cancel{\Delta x}} =$$

$$= \underbrace{\lim_{\Delta x \to 0} 3(2x) + 3(\Delta x)} = 3(2x).$$

This limit is also fairly intuitive to evaluate.

N.B. — See here that for $f(x) = ax^2 + c$, the slope predictor fn. is $\underline{m(x) = 2ax}$.

e.g., For the most general parabola $f(x) = ax^2 + bx + c$:

$$m(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[a(x+\Delta x)^2 + b(x+\Delta x) + c] - [ax^2 + bx + c]}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{ax^2} + 2ax\Delta x + a(\Delta x)^2 + \cancel{bx} + b\Delta x + \cancel{c} - \cancel{ax^2} - \cancel{bx} - \cancel{c}}{\Delta x}$$

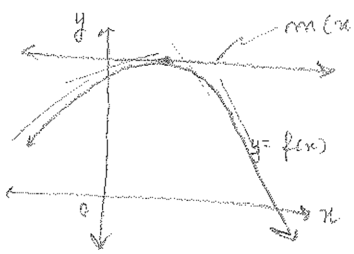
$$= \lim_{\Delta x \to 0} \frac{2ax\cancel{\Delta x} + a(\Delta x)^2 + b\cancel{\Delta x}}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} 2ax + b + \cancel{a\Delta x} = \boxed{2ax + b}$$

0

(This is eq'n (10) on p. 59)

Recall: $m(x)$ is the slope of the line tangent to f at x. (Think of it as the "slope of the function" at x.)

Especially useful — at this point in our studies — for investigating parabolas.



$m(x) = 0 \Rightarrow$ horizontal tangent line

$\Rightarrow$ an extremum (max or min.)

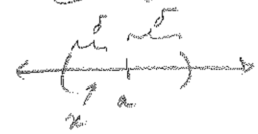
This is only an intuitive observation — it is not a theorem or a rigorous proof (yet)!! — But now, for us, it is good enough ...

More about limits ...

Up until now, the concept of the limit has not been rigorously introduced, and our reasoning abt. limits has been restricted to intuition alone. Let's do this better.

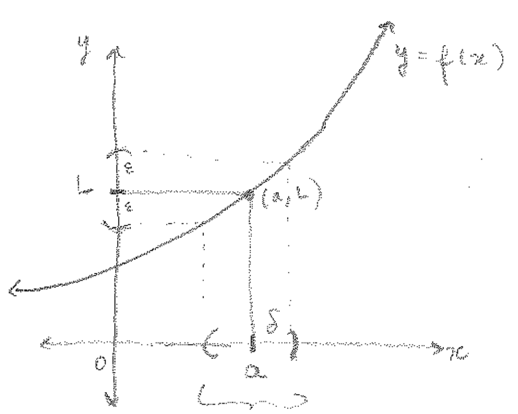
Def. We write $\lim_{x \to a} f(x) = L$ (read "the limit of $f(x)$ as x approaches a is L") when:

$$\underbrace{\overbrace{\forall}^{\text{"for all"}} \varepsilon}_{\text{"epsilon"}} > 0, \quad \overbrace{\exists}^{\text{"there exists a"}} \underbrace{\delta}_{\text{"delta"}} > 0 \quad \overbrace{\text{s.t.}}^{\text{"such that"}} \quad 0 < \underbrace{|a - x| < \delta}_{} \overbrace{\Rightarrow}^{\text{"implies"}} |L - f(x)| < \varepsilon.$$



(usually in mathematics, ε is a small positive quantity)

What does this mean?



everything in here satisfies $|a-x| < \delta$.
To satisfy $0 < |a-x|$, just exclude $x = a$ itself.

Can make $y = f(x)$ arbitrarily close to L just by requiring that x be sufficiently close to a.

$\boxed{\text{What is "sufficient" depends on } f \text{ and on } a.}$

Example. Given $\varepsilon > 0$, find the sufficient $\delta > 0$ s.t.

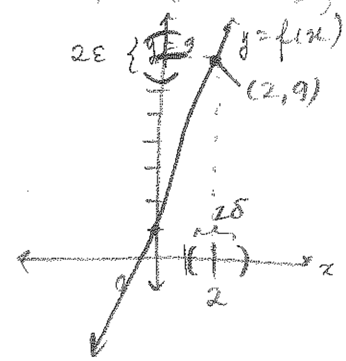
$$0 < |\underbrace{x - 2}_{}| < \delta \Rightarrow |\underbrace{(4x+1)}_{f(x)} - 9| < \varepsilon$$ (N.B.– $f(x) = 4x+1$, $a = 2$, $L = 9$.)

Well, $|(4x+1) - 9| = |4x - 8| = 4|x-2|$, so $|(4x+1)-9| < \varepsilon \Leftrightarrow 4|x-2| < \varepsilon$.

"if and only if" – $a \Leftrightarrow b$ means both $a \Rightarrow b$ (a implies b) AND $b \Rightarrow a$ (b " " a).

But, easy to see that if $|x-2| < \frac{\varepsilon}{4}$, then $4|x-2| < \varepsilon$. So taking $\delta = \frac{\varepsilon}{4}$ is sufficient, i.e.,

$$|x-2| < \frac{\varepsilon}{4} \Rightarrow 4|x-2| < \varepsilon \Rightarrow |\overbrace{(4x+1)}^{f(x)} - 9| < \varepsilon.$$



So, we proved that $\boxed{\lim_{x \to 2} 4x+1 = 9.}$

We now have a way to guarantee that — for example — $f(x) \in (\underbrace{8.99, 9.01}_{\varepsilon = 0.01})$. Just require that $x \in (\underbrace{1.9975, 2.0025}_{\delta = \frac{\varepsilon}{4} = 0.0025})$.

$f(x) = 4x+1$

See link under "Course Materials" on myWPI for a page with more examples of what we call "ε, δ" proofs.

... That's great for proving that a limit is what we said it was (rigorously), but how do we find a limit?

"Normally", SUBSTITUTION works.
(what does this mean?)

$$\lim_{x \to 2} 4x+1 = 4(2)+1 = 9.$$

$$\lim_{x \to 2} x^2+3 = 2^2+3 = 7.$$

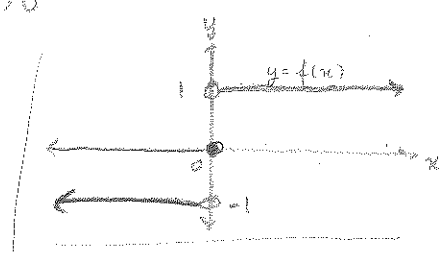
EXCEPTIONS ...

① a is in the domain of the function, but the function is a "nonstandard" one.
(what does this mean?)

$$|x| = \begin{cases} -x, & x<0 \\ x, & x \geq 0 \end{cases}$$

e.g., $f(x) = \begin{cases} x/|x|, & x \neq 0 \\ 0, & x=0 \end{cases} = \begin{cases} -1, & x<0 \\ 0, & x=0 \\ 1, & x>0 \end{cases}$ (saw this fn. last time)

$$\lim_{x \to 0} f(x) = ?$$



If $x \to 0$ from the left (denoted $x \to 0^-$), then $\lim_{x \to 0^-} f(x) = -1$.

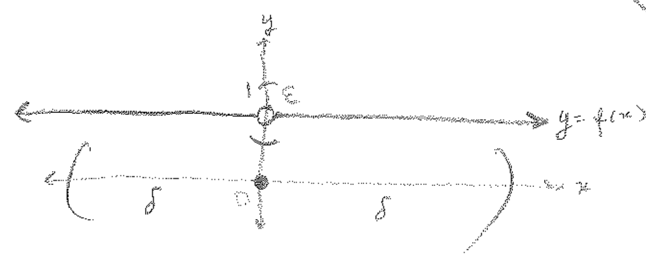
If $x \to 0$ from the right, then $\lim_{x \to 0^+} f(x) = 1$.

The Ⓛ and Ⓡ - hand limits don't agree - so we say that the limit DOES NOT EXIST (D.N.E.).

EXCEPTIONS, CT'D.

(1) ct'd.

e.g., $f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$



$$\lim_{x \to 0} f(x) = ?$$

Note, for all $x \neq 0$ (written another way: $\forall x$ s.t. $0 < |x - 0|$),
↳ why did I write it this way?

have $f(x) = 1$ (i.e., $\underbrace{|f(x) - 1|}_{< \varepsilon \ \forall \varepsilon > 0} = 0$).

So, for this function any δ is sufficient! — Pick $\delta = 1$.

Then

$$\forall \varepsilon > 0, \quad 0 < \underbrace{|x - 0| < 1}_{\text{our } \delta \text{ choice}} \Rightarrow \underbrace{|f(x) - 1|}_{= 0 \text{ — saw this above}} < \varepsilon.$$

could have said $\delta = 1000$ or $\delta = 0.01$ or $\delta = \varepsilon$ ... same result

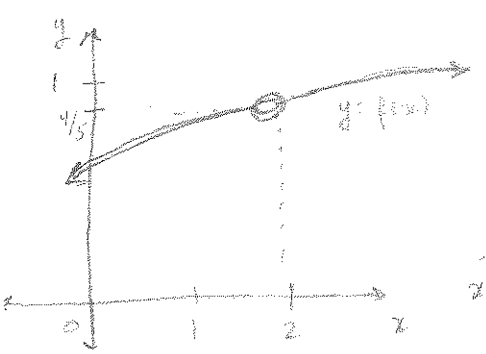
So, $\lim_{x \to 0} f(x) = 1$.

But $f(0) = 0$, so simple substitution wouldn't have worked.

(2) a might not be in the domain of f.

(#2, p.64

e.g., $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + x - 6}$.   Note, denominator is zero at $x = 2$,
so 2 not in domain of f, BUT ...



The graph suggests that the limit is $\frac{4}{5}$.
Observe:

$$\lim_{x \to 2} \frac{x^2 - 4}{x^2 + x - 6} = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(x+3)} = \lim_{x \to 2} \frac{x+2}{x+3} = \frac{4}{5}.$$

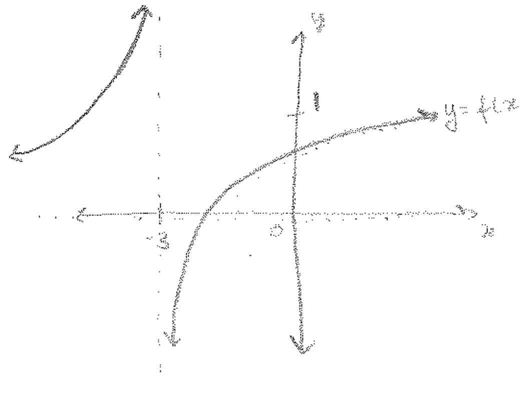
$x \neq 2$ "removable discontinuity"

SUBSTITUTION

example 2, p. 66 ct'd. $\lim_{x \to -3} \frac{x^2-4}{x^2+x-6} = ?$

Same fn. as before, but can't cancel in the denominator:

$$\lim_{x \to -3} \frac{x^2-4}{x^2+x-6} = \lim_{x \to -3} \frac{(x+2)(x-2)}{(x-2)(x+3)} = \lim_{x \to -3} \frac{x+2}{x+3}.$$



Looking at graph, see

$\lim_{x \to -3^-} f(x) = +\infty$

BUT $\lim_{x \to -3^+} f(x) = -\infty$.

"non-removable discontinuity" $x=-3$.

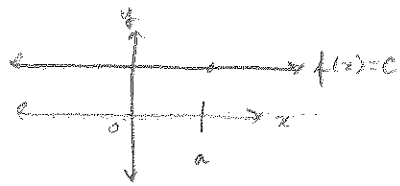
These don't agree, so limit does not exist.

MORAL OF THE STORY ... always check the graph!

always factor rational expressions, to see if discontinuity is removable (get to this again later)

Limit Rules.

Some easy limits:



CONST. $\lim_{x \to a} C = C$ ($f(x) = C$ is a constant)

SUM. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist,

Then

$$\lim_{x \to a} \left[f(x) \pm g(x) \right] = L \pm M.$$

PRODUCT. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist,

Then $\lim_{x \to a} \left[f(x) g(x) \right] = LM.$

QUOTIENT. If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$ both exist AND $M \neq 0$,

Then $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{M}.$

ROOT. If $n \in \overbrace{\mathbb{Z}^+}^{\text{"is a positive integer"}}$ and if $a > 0$ for even values of n,

Then $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}.$

$$\boxed{\begin{array}{l} \lim_{x \to a} x = a. \\ \lim_{x \to a} x^m = a^m \end{array}}$$

NOTE: $\lim_{x \to a} x^{m/n} = a^{m/n}$, $m > 0$ and $a > 0$ if n is even.

POLYNOMIALS.

$$\lim_{x \to a} \left[c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \right] =$$

$$= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0$$

<u>can do</u> by substitution!

COMPOSITION. If $\lim_{x \to a} g(x) = L$

AND

$\lim_{x \to L} f(x) = M$

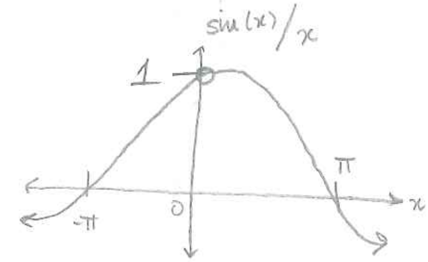
AND

$M = f(L)$ — must be able to obtain by substitution (continuity ... a preview)

Then $\lim_{x \to a} f(g(x)) = M$.

TRIGONOMETRIC.

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$



Other: $\lim_{x \to 0} \cos(x) = 1$ and $\lim_{x \to 0} \sin(x) = 0$ (substitution)

$(1+a)(1-a) = 1 \cancel{-a+a} - a^2 = 1 - a^2$

e.g., #1, p.77

$$\underbrace{\lim_{x \to 0} \frac{1-\cos x}{x}}_{=0.} = \lim_{x \to 0} \left(\frac{1-\cos x}{x}\right)\underbrace{\left(\frac{1+\cos x}{1+\cos x}\right)}_{=1} = \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)} =$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{1+\cos x}\right) =$$

equal ONLY IF both exist.

$$\boxed{\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \quad \forall x. \\ 1 - \cos^2(x) &= \sin^2(x) \quad \forall x. \end{aligned}}$$

$$\lim_{x \to 0} \underbrace{\left(\frac{\sin x}{x}\right)}_{1}\underbrace{\left(\frac{\sin x}{1+\cos x}\right)}_{0}$$

1

$$\frac{\sin(0) \to 0}{1+\cos(0)} \quad \cos(0) \to 1 \qquad \frac{0}{2} = 0$$

SQUEEZE THEOREM.

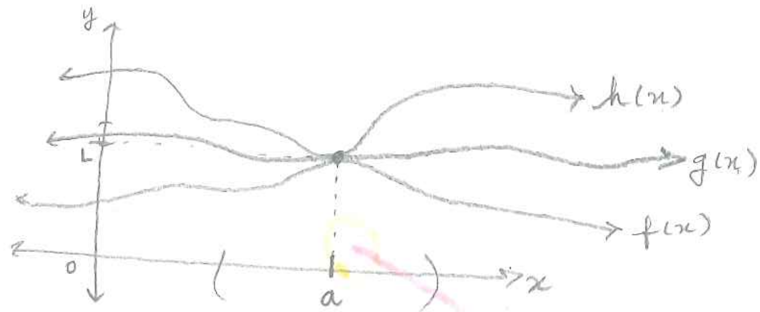
"neighborhood" = "interval around"

If $f(x) \le g(x) \le h(x)$ $\forall x \ne a$ in some nbd. of a,

and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$,

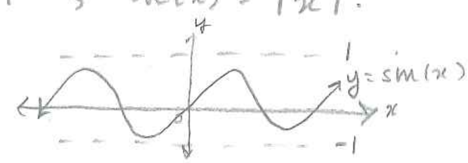
Then $\lim_{x \to a} g(x) = L$.

Picture:



ex. #4, p.78

$g(x) = x \sin\left(\frac{1}{x}\right)$. Take $f(x) = -|x|$, $h(x) = |x|$.



Observe: $-|x| \le x \underbrace{\sin\left(\frac{1}{x}\right)}$

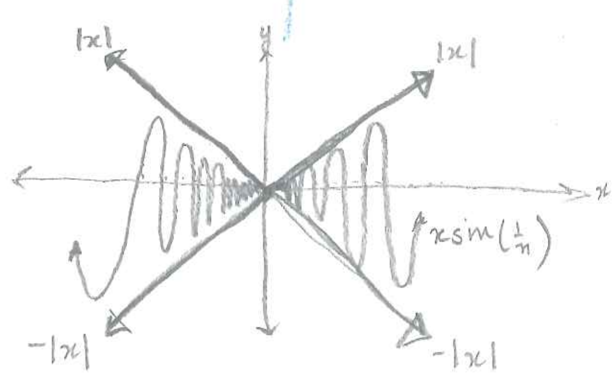
bounded above & below by ± 1

and $x \sin\left(\frac{1}{x}\right) \le |x|$.

Note also, $\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$.

By the squeeze theorem, $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$ as well.

See:



Recall: (L) and (R)-hand limits.

Suppose f is defined for $x \neq a$ in a $\overset{\text{neighborhood}}{\widehat{\text{nbd.}}}$ of a.

"deleted nbd. of a"

Then

$\lim\limits_{x \to a} f(x)$ exists and is equal to L

$\iff$ (if and only if)

$\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a^-} f(x)$ both exist and $\underset{\text{both}}{\wedge}$ equal L.

"if and only if"

What $\overset{}{\widehat{\text{iff}}}$ means: $a \iff b$ is a powerful statement. It means ALL of the following:

- $a \Rightarrow b$
- $b \Rightarrow a$
- NOT $b \Rightarrow$ NOT a
- NOT $a \Rightarrow$ NOT b.

So if we want to show that $\lim\limits_{x \to a} f(x)$ D.N.E., then we could show that $\lim\limits_{x \to a^+} f(x) \neq \lim\limits_{x \to a^-} f(x)$. (or, we could show that either $\lim\limits_{x \to a^+} f(x)$ or $\lim\limits_{x \to a^-} f(x)$ D.N.E.)

If we want to show that $\lim\limits_{x \to a} f(x) = L$, then we could show that $\lim\limits_{x \to a^+} f(x) = L = \lim\limits_{x \to a^-} f(x)$.