Lecture 3: 28 May 2014.

TODAY: Finish up Chapter 2 (CONTINUITY, IVT)
Sec'ns 3.1 (DERIVATIVES & RATES OF CHANGE)
3.2 (DIFFERENTIATION RULES)

Continuity. In the last class, we noticed that for some functions and some points, we can compute the limit simply by substitution; that is, for some $f(x)$ and some $a \in \mathbb{R}$, we have

$$\lim_{x \to a} f(x) = f(a).$$

If, for some f and some a, this property holds, we say: f is $\underbrace{\text{continuous}}_{\text{sometimes abbreviated "cts."}}$ at a.

When we say f is continuous on a set S, then we mean that for each element a of S, f is continuous at a.
(e.g., f is cts. on the real line $\Rightarrow S = \mathbb{R}$)

We found several functions in the last lecture that were cts. on the entire real line $\mathbb{R}$:

- Constant functions $f(x) = c$, as $\lim_{x \to a} c = c = f(a)$
- Linear functions e.g. $f(x) = x$, as $\lim_{x \to a} x = a = f(x)$
- Polynomials $f(x) = c_n x^n + \cdots + c_1 x + c_0$, by the multiplication and addition rules for limits.

More on cts. functions.

We also saw two different kinds of discontinuities last time (we called them "removable" and "non-removable"). Let's look at a third kind, and give proper names for all three.

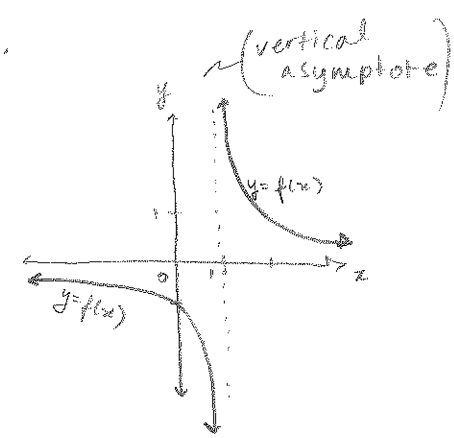
FIRST: Note that for a fn. f to be cts. at a, it must satisfy three conditions:

1. f must be defined at a (so $f(a)$ exists)
2. $\lim_{x \to a} f(x)$ must exist
3. Must have $\lim_{x \to a} f(x) = f(a)$.

If any of these conditions fails, then f is discontinuous at a, or a is a discontinuity of f.

WHEN ① FAILS

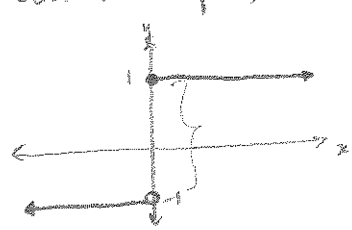
Consider $f(x) = \dfrac{1}{x-1}$. The graph:



At $x = 1$, f has what we called a "non-removable" discontinuity (cannot cancel the $(x-1)$ term from denominator). Also called an INFINITE DISCONTINUITY.

WHEN ② FAILS

Consider $f(x) = \text{sign}(x) = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases}$. We noted that the left- and right-hand limits were not equal at 0:

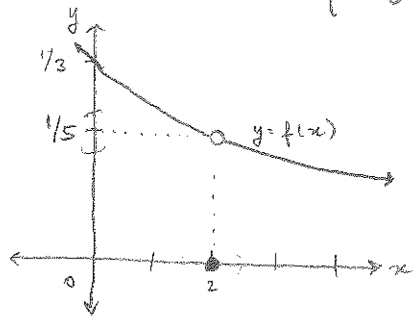


$\lim_{x \to 0^+} = 1 \neq \lim_{x \to 0^-} = -1$. So the limit DNE.

This is called a FINITE JUMP DISCONTINUITY.

WHEN ③ FAILS

Consider $f(x) = \begin{cases} \dfrac{x-2}{x^2+x-6}, & x \neq 2 \\ 0, & x = 2 \end{cases} = \begin{cases} \dfrac{x-2}{(x-2)(x+3)}, & x \neq 2 \\ 0, & x = 2. \end{cases}$



$g(x) := \dfrac{x-2}{(x-2)(x+3)}$ has what we called a "removable" discontinuity at $x=2$.

However, our definition of $f(x)$ did not remove the discontinuity, since $\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = 1/5$, but the way we defined f, $f(2) = 0$.

To successfully "remove" the discontinuity of g at $x=2$, we want to define a new function $h(x)$ so that:

- $h(x) = g(x)$ for all $x \neq 2$ in the domain of g

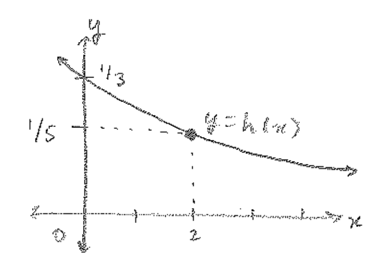
  (recall: $D_g = \{x \in \mathbb{R} : x \neq 2 \text{ and } x \neq -3\}$)

- h is cts. at $x=2$.

  $\hookrightarrow$ (n.b.: $\lim_{x \to 2} g(x) = 1/5$, so want $h(2) = 1/5$).

Easy: $h(x) = \begin{cases} g(x), & x \neq 2 \\ 1/5, & x = 2 \end{cases}$

$= \begin{cases} \dfrac{x-2}{(x-2)(x+3)}, & x \neq 2 \\ 1/5, & x = 2 \end{cases}$



Cts. fns.

The limit rules (mult./division, add'n/subtrac'n) give way to some handy rules abt. continuity as well:

Suppose $f(x)$ and $g(x)$ are both cts. at $x=a$.

Then $\underbrace{f(a) \text{ exists}, \; g(a) \text{ exists}}_{1}$, $\underbrace{\lim_{x\to a} f(x) \text{ exists}, \; \lim_{x\to a} g(x) \text{ exists}}_{2}$, and

(*) $\lim_{x\to a} f(x) = f(a)$ and (**) $\lim_{x\to a} g(x) = g(a)$, all by the def'n of continuity.

Now, by computation: $(f+g)(a) = f(a) + g(a)$.

and $\lim_{x\to a} (f+g)(x) = \underbrace{\lim_{x\to a} f(x)} + \underbrace{\lim_{x\to a} g(x)}$

limit law for add'n of fns.

$= f(a) + g(a)$.

(*) and (**)

Thus, $\lim_{x\to a} (f+g)(x) = (f+g)(a)$, so $f+g$ is cts. at $x=a$ as well.

CONTINUITY RULES: If $f(x)$ and $g(x)$ are both cts. at $x=a$, Then:

1. $(f+g)(x)$
2. $(f-g)(x)$
3. $(fg)(x)$
4. $f(g(x)) = (f\circ g)(x)$

are all cts. at $x=a$, too.

Additionally, if $g(a) \neq 0$, then 5. $\left(\frac{f}{g}\right)(x)$ is also cts. at $x=a$.

Special note on the division case:

If $g(a)=0$, then two cases: either

① $f(a)\neq 0$, so $\frac{f}{g}$ has an INFINITE DISCONTINUITY at $x=a$;

or ② $f(a)=0$, and $\frac{f}{g}$ <u>may</u> have a REMOVABLE DISCONTINUITY at a.

<u>examples.</u> $a=2$

$f(x)=x$, $g(x)=x^2-3x+2$. Then $g(2)=0$ but $f(2)\neq 0$ (case ①)

$f(x)=x-2$, $g(x)=x^2-3x+2$. Then $f(2)=0$, so investigate further:

$$\frac{f}{g}(x)=\frac{x-2}{x^2-3x+2}=\frac{\cancel{x-2}}{\cancel{(x-2)}(x-1)}=\frac{1}{x-1}$$, so discontinuity

was removable.

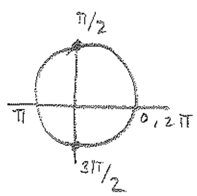
$f(x)=e^x-e^2$, $g(x)=x^2-3x+2$. Then $f(2)=0$, so investigate:

$$\frac{f}{g}(x)=\frac{e^x-e^2}{x^2-3x+2}$$. Cannot cancel; discontinuity is not removable.

<u>Other cts. fns.:</u> $f(x)=\sin(x)$ and $f(x)=\cos(x)$ are cts. on $\mathbb{R}$.

$g(x)=\tan(x)=\frac{\sin(x)}{\cos(x)}$ is cts. wherever $\cos(x)\neq 0$;

i.e., cts. on $S=\{x\in\mathbb{R} : x\neq \underbrace{\frac{(2m+1)\pi}{2}}_{\pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \dots},\ m\in\mathbb{Z}\}$.



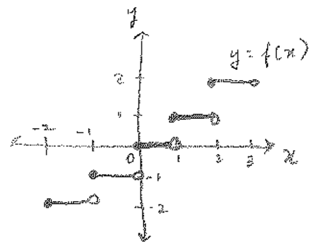
$\cos\theta=\frac{x}{r}$

Left- and Right-continuity.

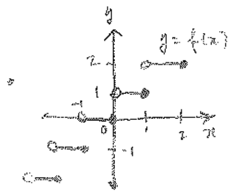
f is left-cts. at $x=a$ if $\lim\limits_{x\to a^-} f(x) = f(a)$

" right-cts. " " $\lim\limits_{x\to a^+} f(x) = f(a)$.

e.g., • $f(x) = \text{floor}(x)$

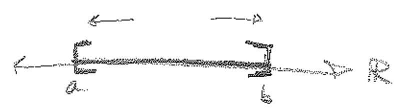


is right cts. at $x = 1$, (and at all other integers) but is not left cts. there.



• $f(x) = \text{ceil}(x)$ is left cts. at $x = 1$ (and at all other integers), but is not right cts.

Continuity on closed intervals.



Say f is cts. on the closed interval $[a,b]$ if:

1. f is cts. at each point $x \in (a,b)$
2. f is right cts. at a
3. f is left cts. at b.

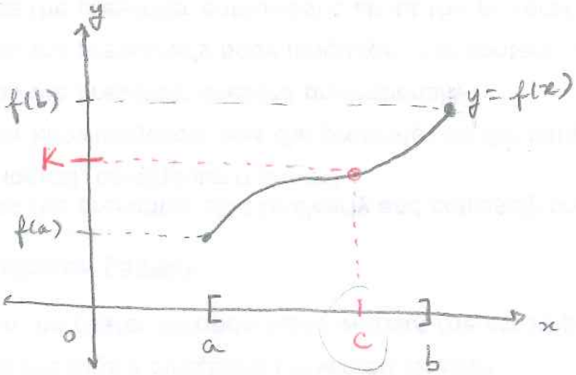
This is a very handy property for a function to have!

- f assumes its min/max. on $[a,b]$ (PROOF LATER)
- INTERMEDIATE VALUE THM.

IVT. (a theorem!) If f is cts. on the closed interval $[a, b]$,
Then f assumes <u>every</u> intermediate value between $f(a)$ and $f(b)$.

Important interpretation: If K is a number btwn. $f(a)$ and $f(b)$, then there is at least one $c \in [a, b]$ s.t. $f(c) = K$.

$\in$ — in (a set, interval)



This is useful!

e.g., Prove that there is at least one solution x to the eq'n $x = \cos(x)$, between 0 and $\pi/2$.

Let $f(x) = x - \cos(x)$. Since $g_1(x) = x$ and $g_2(x) = \cos(x)$ are both cts. everywhere in $\mathbb{R}$, then their difference $f(x) = x - \cos(x)$ is also ——— " ————————, in particular on the closed interval $[0, \pi/2]$. Note that $f(0) = 0 - \cos(0) = -1$, and $f(\pi/2) = \frac{\pi}{2} - \cos(\pi/2) = \pi/2$. Since $-1 < 0 < \frac{\pi}{2}$, we can apply the IVT to f on $[0, \pi/2]$, with the result that there is at least one $x \in [0, \pi/2]$ s.t. $f(x) = 0$ — that is, $x - \cos(x) = 0$, or $x = \cos(x)$.

THE DERIVATIVE.

Recall the "slope predictor formula" — it gives the slope of the line tangent to a fn. $f(x)$ at the point a:

$$m(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}.$$

Let's make a function out of this, and let's call this function the DERIVATIVE of f:

$$f'(x) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Note: the limit is taken holding x fixed, and varying only Δx.

If this limit exists for $x=a$, then we say f is DIFFERENTIABLE at $x=a$.

Recall: We found the slope-predictor function for a general quadratic:

$f(x) = ax^2 + bx + c \quad \Longrightarrow$ (implies)

$$f'(x) = \lim_{\Delta x \to 0} \frac{a(x+\Delta x)^2 + b(x+\Delta x) + c - ax^2 - bx - c}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{ax^2} + 2ax\Delta x + (\Delta x)^2 + \cancel{bx} + b\Delta x \cancel{+c} \cancel{-ax^2} - \cancel{bx} \cancel{-c}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^{\cancel{2}} + (2ax+b)\cancel{\Delta x}}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \underbrace{\cancel{\Delta x}}_{\to 0} + \underbrace{2ax+b}_{\text{"}c\text{"}}$$

$$= 2ax + b.$$

Also recall: Can use the slope predictor to write the eq'n of a tangent line:

$$y - f(a) = f'(a)\,(x - a)$$

SLOPE OF TAN LINE

POINT $(a, f(a))$ IS ON THE TAN LINE

POINT - SLOPE FORM

$$y - y_0 = m(x - x_0)$$

(x_0, y_0) is on the line

m is the slope.

example: $f(x) = 3x^2 - 4x + 5$. Write the eq'n of the tan. line at $x = 2$.

Note, $f(2) = 3(2^2) - 4(2) + 5 = 12 - 8 + 5 = 9$.

$f'(x) = 2(3)x + (-4) = 6x - 4$, so $f'(2) = 6(2) - 4 = 12 - 4 = 8$.

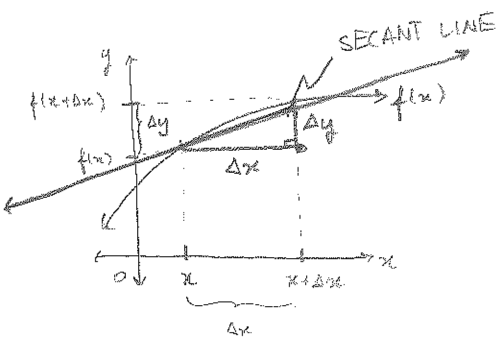
Thus, the eq'n of the tan. line is:

$$y - 8 = 9(x - 2), \quad \text{or} \quad y = 9x - 10.$$

Differential notation.

If Δx is the change in x, then the change in y over this distance is:

$$\Delta y = f(x + \Delta x) - f(x).$$



Then the slope of the secant line connecting $(x, f(x))$ with $(x + \Delta x, f(x + \Delta x))$ is:

$$m_s = \frac{\Delta y}{\Delta x} \quad \text{(see graph).}$$

"secant" ← a quotient

As we did before, to obtain the slope of the tangent line, we let $\Delta x \to 0$. This is denoted:

$y = f(x)$ $\boxed{m_T = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \doteq \frac{dy}{dx}}$, so write $\boxed{f'(x) = \frac{dy}{dx}.}$

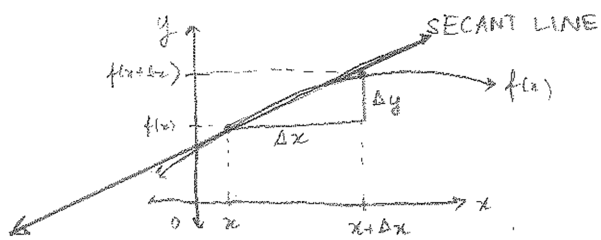
* NOT A QUOTIENT *

We discuss "differential" notation in Ch. 4, but important to remember now is that the independent variable is "downstairs" and the dependent variable is "upstairs", e.g.,

$$\frac{dy}{dx}, \quad \frac{dv}{dt}, \quad \text{etc.}$$

RATES OF CHANGE.

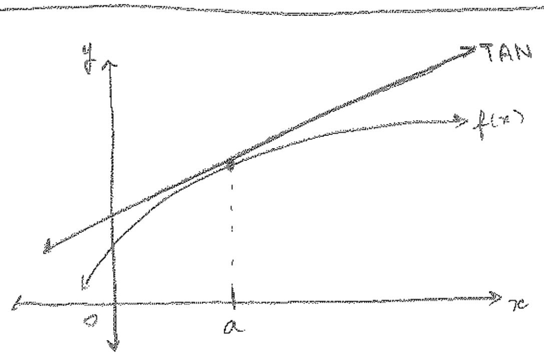
We discussed the idea that the slope of the secant line tells us the rate of change of the fn. over an interval:



(The slope of f over $(x, x+\Delta x)$ is $\frac{\Delta y}{\Delta x}$.)

Extension of this idea to tangent lines:

The slope of the line tangent to a fn. at a point is the INSTANTANEOUS RATE OF CHANGE of the fn. AT THAT POINT:



The instantaneous rate of change of f at $x=a$ is $\left.\frac{dy}{dx}\right|_{x=a} = f'(a)$.

(the vertical bar with subscript means "this expression evaluated at the subscript")

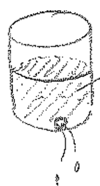
- Positive slope = "rising" line

$\Rightarrow$ $f(x)$ is INCREASING if $f'(x) > 0$.

- Negative slope = "falling" line

$\Rightarrow$ $f(x)$ is DECREASING if $f'(x) < 0$.

Ex. (3, p. 111)



$V(0) = 600$ gal ; $V(\overset{\text{min}}{60}) = 0$.

$V(t) = \frac{1}{6}(60-t)^2 = 600 - 20t + \frac{1}{6}t^2$.

Find instantaneous rate of change at $t = 15$ min, $t = 45$ min, and find avg. rate of change during the half-hour from $t = 15$ to $t = 45$ min.

Sol'n: Know that for $V(t) = at^2 + bt + c$, then

$$\frac{dV}{dt} = V'(t) = 2at + b.$$

So for us, $\frac{dV}{dt} = V'(t) = 2\frac{1}{6}t - 20 = \frac{1}{3}t - 20$.

The inst. rate of change of V at $t = 15$ is given by

$$\left.\frac{dV}{dt}\right|_{t=15} = V'(15) = \frac{1}{3}(15) - 20 = -15 \text{ gal/min}$$

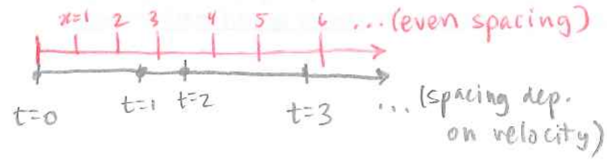
and at 45 min: $\left.\frac{dV}{dt}\right|_{t=45} = V'(45) = \frac{1}{3}(45) - 20 = -5$ gal/min.

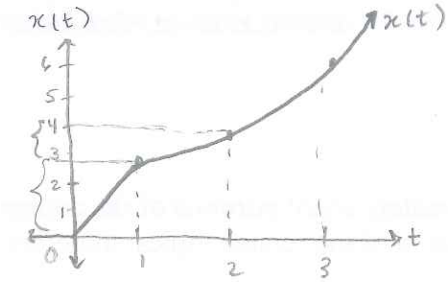
The avg. rate of change over (15, 45):

$$\left.\frac{\Delta V}{\Delta t}\right|_{(15,45)} = \frac{V(45) - V(15)}{45 - 15} = \frac{\frac{1}{6}(600-45)^2 - \frac{1}{6}(600-15)^2}{45-15} = \frac{-300}{30} = -10 \frac{\text{gal}}{\text{min}}$$

Velocity, acceleration.

Suppose $x(t)$ is the position function of an object (a car, a bumblebee, etc.) travelling along a horizontal straight line:





average velocity over a time interval = slope of secant line of $x(t)$ over that interval $= \dfrac{\Delta x}{\Delta t} = \dfrac{x(t+\Delta t) - x(t)}{\Delta t}$

$\Delta t \to 0$

instantaneous velocity at a point in time = slope of line tangent to $x(t)$ at that time $= \dfrac{dx}{dt} = x'(t)$.

"VELOCITY IS THE INSTANTANEOUS RATE OF CHANGE OF POSITION"

Similar for acceleration – it is the inst. rate of change of the velocity.

—"— "jerk factor" – —"— of the acceleration.

$f(t) = at^2+bt+c \Rightarrow f'(t) = 2at + b$

in feet

ex. (5, p. 113)

$x(t) = 5t^2 + 100$. Then $v(t) = x'(t) = 2(5)t = 10t$.

So $v(0) = 10(0) = 0$ ⇒ car starts at rest

$v(10) = 10(10) = 100$ f.p.s.

$x(0) = 5(0) + 100 = 100$ ft. ⇒ car starts at $x = 100$

$x(10) = 5(10^2) + 100 = 600$ ft.

So avg. velocity over $t = (0, 10)$ is $\dfrac{\Delta x}{\Delta t} = \dfrac{600 - 100}{10 - 0} = 50$ fps.

But inst. velocity at $t = 10$ is $\left.\dfrac{dx}{dt}\right|_{t=10} = x'(10) = v(10) = 100$ f.p.s.

Vertical motion is similar! — See p. 114, ex. 6.

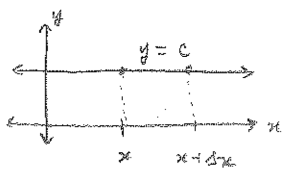
DIFFERENTIATION RULES.

So, we have these neat characterizations of the derivative — and it could be really useful — if we knew how to compute them without taking big, complicated limits all the time!

Thankfully, we have a few handy rules.

(1) If $f(x) = c$ constant, then $f'(x) = 0$.

Proof:



tan. line has slope zero!
(In fact, tan. line is c itself)

(2) If $f(x) = x^n$, then $f'(x) = n x^{n-1}$. "POWER RULE" (for $n \in \mathbb{N}$) $n = 1, 2, 3, 4, \ldots$

Proof: limit laws, p. 121 of text.

(3) LINEAR COMBINATION: If $f(x) = a\,g(x) + b\,h(x)$, $a, b \in \mathbb{R}$ consts.,

Then $f'(x) = a\,g'(x) + b\,h'(x)$.

(Note that two handy sub-rules follow immediately:

$b = 0$: CONSTANT MULTIPLES RULE

$$\frac{d}{dx}\left[a\,g(x)\right] = a\frac{d}{dx}g(x) \qquad \frac{dy}{dx} = \frac{d[a\,g(x)]}{dx}$$

$a = b = 1$: ADDITION RULE

$$\frac{d}{dx}\left[g(x) + h(x)\right] = \frac{d}{dx}\left[g(x)\right] + \frac{d}{dx}\left[h(x)\right].$$

(2) and (3) combine to give us the POLYNOMIAL RULE:

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$$

$$\Rightarrow \quad \text{(implies)}$$

$$f'(x) = m \cdot a_m x^{m-1} + (m-1) \cdot a_{m-1} x^{m-2} + \cdots + 3a_3 x^2 + 2a_2 x + a_1 .$$

ex. $f(x) = 3x^5 + 9x^4 + 1 \cdot x^2 + 17x + 31$

$$\Rightarrow$$

$$f'(x) = 5(3)x^4 + 4(9)x^3 + 2(1)x + 17$$

$$= 15x^4 + 36x^3 + 2x + 17 .$$

ex. (5, p. 123)

$$V(T) = V_0\left[1 - \overbrace{(6.427 \times 10^{-5})}^{a_1} T + \overbrace{(8.505 \times 10^{-6})}^{a_2} T^2 - \overbrace{(6.790 \times 10^{-8})}^{a_3} T^3\right],$$

where $V_0 = \underbrace{10^5}_{=100,000}$ cm³ is constant. (Note: [V] = cm³, [T] = °C)

Find volume and rate of change of volume $\underbrace{\text{with respect to}}_{\text{"w.r.t."}}$ temp. when T = 20°C.

$$V(20°C) = 10^5\left[1 - (6.427 \times 10^{-5})(20) + (8.505 \times 10^{-6})(20^2) - (6.790 \times 10^{-8})(20^3)\right]$$

$$\approx 100,157.34$$

$$V'(T) = V_0\left[-6.427 \times 10^{-5} + 2(8.505 \times 10^{-6})T - 3(6.790 \times 10^{-8})T^2\right]$$

so $V'(20°C) = V_0\left[-6.427 \times 10^{-5} + 2(8.505 \times 10^{-6})(20) - 3(6.790 \times 10^{-8})(20^2)\right]$

$$\approx 19.45 \text{ cm}^3/°C .$$

$$V(T) = V_0\left[1 - a_1 T + a_2 T^2 - a_3 T^3\right]$$

$$V'(T) = V_0\left[0 - a_1 + 2a_2 T - 3a_3 T^2\right]$$

(4) PRODUCT RULE. f, g - diff'ble at x $\Rightarrow$ fg also diff'ble at x, (differentiable)
and

$$\frac{d}{dx}[fg] = f\frac{d}{dx}g + g\frac{d}{dx}f$$

$$= f\,g'(x) + g\,f'(x).$$

or $(fg)' = fg' + gf'$.

Proof: p. 124.

Ex. Find the derivative of $f(x) = (3x^2+7)(1-2x)$.

Hard way: $f(x) = 3x^2 - 6x^3 + 7 - 14x$

$\Rightarrow f'(x) = 6x - 18x^2 - 14$

Easy way: $f'(x) = (3x^2+7)\frac{d}{dx}[1-2x] + (1-2x)\frac{d}{dx}[3x^2+7]$

$= (3x^2+7)(-2) + (1-2x)(6x)$

$= -6x^2 - 14 + 6x - 12x^2$

$= -18x^2 + 6x - 14$.

⑤ QUOTIENT RULE.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

If we use D_x as a $\underbrace{\text{"differential operator"}}_{\text{i.e., } \frac{d}{dx}[\cdots] \doteq D_x[\cdots]}$ notation, then

can write

$$\underbrace{D_x\left[\frac{Hi}{Ho}\right] = \frac{Ho\ D_x Hi - Hi\ D_x Ho}{Ho^2}}_{\text{this is just a mnemonic – ignore if confusing}}.$$

Or in another form:

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}.$$

Example. Suppose $z(t) = \frac{1-t^3}{1+t^4}$. Then, using the formula,

$$z'(t) = \frac{\frac{d}{dt}[1-t^3](1+t^4) - \frac{d}{dt}[1+t^4](1-t^3)}{(1+t^4)^2}$$

$$= \frac{(-3t^2)(1+t^4) - (4t^3)(1-t^3)}{(1+t^4)^2}$$

$$= \frac{t^6 - 4t^3 - 3t^2}{(1+t^4)^2}$$