Lecture 4: 02 June.

TODAY: 3.7 Derivatives of Trig. Functions
3.8 — " — Exponential & Logarithmic Functions
3.3 The chain rule.

3.3: The Chain Rule

Last time, we went over some very useful rules for differentiation:

1. Constant rule: $\frac{d}{dx}[c] = 0$

2. Power rule: $\frac{d}{dx}[x^n] = n x^{n-1}$, for $n \in \mathbb{N}, \mathbb{Q}, \mathbb{Z}$
(recall: $\mathbb{N} = \{1, 2, 3, \dots\}$.)

3. Linear combination Rule:
$$\frac{d}{dx}\left[a f(x) + b g(x)\right] = a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)]$$
$$= a f'(x) + b g'(x)$$
recall: "prime" notat'n vs. differential ("d") notation

4. Product rule:
$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x).$$

5. Quotient rule:
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

This time, we add an extremely useful rule:

6. THE CHAIN RULE:
$$\frac{d}{dx}\left[\underbrace{f(g(x))}\right] = f'(g(x))\,g'(x).$$

function composition: $f \circ g$ is another notation

For example, let's let $g := g(x)$ be an arbitrary function, and let's let $f(u) := u^2$. Then $f(g(x)) = [g(x)]^2$, and we know how to differentiate this via the product rule already:

$$\frac{d}{dx}\left[f(g(x))\right] = \frac{d}{dx}\left[[g(x)]^2\right] = \frac{d}{dx}\left[g(x)\,g(x)\right]$$

$$= g(x)\,g'(x) + g'(x)\,g(x)$$

$$= 2\,g(x)\,g'(x)$$

Q. Is this a surprise?

... we already knew from the polynomial rule that $\frac{d}{du}[u^2] = 2u$, so if we'd naively guessed that $\frac{d}{dx}\left[[g(x)]^2\right] = 2g(x)$, then we would have been <u>wrong</u>!

But let's use what we found:

We know $f'(u) = 2u$, so $\boxed{f'(g(x)) = 2g(x).}$

We found $\frac{d}{dx}\left[f(g(x))\right] = \boxed{2\,g(x)}\,g'(x)$

$$= f'(g(x))\,g'(x) \quad \text{(plugged it in).}$$

As it turns out, this formula is valid with <u>any</u> two functions $f(x)$ and $g(x)$. (I wrote $f := f(u)$ in order to keep notation more clear... See page 131 of text, where they write

$y := y(u)$, $u := u(x)$ and obtain the DIFFERENTIAL FORM:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}}$$

<u>Recall</u>: differentials are <u>NOT</u> quotients, so "cancelling" does NOT work!

c.f.: $\frac{\sin x}{n} = \sin x = \sin = 6$

Using the Chain Rule.

Ex. Suppose $y = (2x^3+4)^{15}$. Find $\frac{dy}{dx}$.

(Note: this would be really difficult with the product rule!)

See: If we let $f(x) = x^{15}$ and $g(x) = 2x^3+4$, then

$y = f(g(x))$, and $\frac{dy}{dx} = \frac{d}{dx}\left[f(g(x))\right]$

$= f'(g(x))\,g'(x)$ by the chain rule.

So we find $f'(x)$ and $g'(x)$ to plug in.

$f'(x) = 15x^{14}$ by the power rule,

so $f'(g(x)) = \underline{15(2x^3+4)^{14}}$.

$g'(x) = 2(3x^2) = \underline{6x^2}$ by the polynomial rule.

So: $\frac{dy}{dx} = f'(g(x))\,g'(x) = 15(2x^3+4)^{14} \cdot 6x^2$

$= 90(2x^3+4)^{14}x^2$.

Ex. $y = \frac{1}{(3x^7-x+1)^2}$. Could do using the quotient rule:

$$\frac{dy}{dx} = \frac{(3x^7-x+1)^2 \cancelto{0}{\frac{d}{dx}[1]} - 1\,\frac{d}{dx}\left[(3x^7-x+1)^2\right]}{(3x^7-x+1)^4} = \frac{-\frac{d}{dx}\left[9x^{14} - 6x^8 + 6x^7 + x^2 - 2x + 1\right]}{(3x^7-x+1)^4}$$

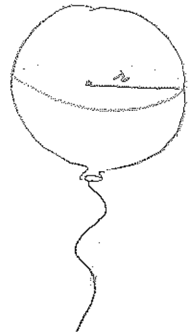
$$= \frac{-\left[9\cdot 14x^{13} - 48x^7 + 42x^6 + 2x - 2\right]}{(3x^7-x+1)^4} = \frac{-126x^{13} + 48x^7 - 42x^6 - 2x + 2}{(3x^7-x+1)^4}$$

OR, with the chain rule. $f(x) = x^{-2}$, $g(x) = 3x^7 - x + 1$, so $f'(x) = -2x^{-3}$ by the power rule, $f'(g(x)) = \underline{-2(3x^7-x+1)^{-3}}$, and $g'(x) = 3(7)x^6 - 1 = \underline{21x^6 - 1}$,

so $\frac{dy}{dx} = \left[-2(3x^7-x+1)^{-3}\right]\left[21x^6-1\right] = \frac{-2(21x^6-1)}{(3x^7-x+1)^3}$.

(CHECK THAT THESE ARE EQUAL)

Ex.
6, p.135



Balloon

At time $t = T$ when $r(T) = 5$ cm

~~When~~ $r = 5$ cm, the radius r of the balloon is increasing at the rate of $0.2\ \frac{\text{cm}}{\text{sec}}$.

At what rate is the volume increasing at that instant?

What we know: $V(r) = \frac{4}{3}\pi r^3$

and although we don't know $r(t)$, we do know that $r'(5) = 0.2$ cm/sec.

What we want to know: $\left.\frac{dV}{dt}\right|_{t=T}$
such that $r(T) = 5$ cm

Can use the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\,\frac{dr}{dt}$$, and know how to find $\frac{dV}{dr}$:

$$\frac{dV}{dr} = \frac{d}{dr}\left[\frac{4}{3}\pi r^3\right] = \frac{4}{3}\pi\,(3r^2) = 4\pi r^2.$$

So $$\left.\frac{dV}{dr}\right|_{t=T} = 4\pi\,(r(T))^2 = 4\pi(5)^2 = 100\pi\ \text{cm}^3/\text{cm}$$

and we were given $\left.\frac{dr}{dt}\right|_{t=T} = 0.2$ cm/sec,

So plug in: $$\left.\frac{dV}{dt}\right|_{t=T} = \underbrace{100\pi}_{\frac{dV}{dr}}\ \underbrace{(0.2)}_{\frac{dr}{dt}} = 20\pi\ \frac{\text{cm}^3}{\text{sec}}$$

≈ 62.83 cm^3/sec.

What this balloon example showed ...

We might not know f and g explicitly, but we might still be able to work with them to get the answer we wanted.

Ex. Conceptest Questions #1, p. 137

You are on a mathematical quiz show, and are asked to calculate $F'(7)$ for $F = f \circ g = f(g(x))$. The functions f and g are unknown, but you are allowed to ask three questions regarding the numerical values of f, g, f', and g' at any points you choose. Which three questions should you ask ??

Well, $F'(x) = f'(g(x))\,g'(x)$, so $F'(7) = f'(\underline{g(7)})\,g'(7)$

So we need to know: ① $f'(g(7))$ and ② $g'(7)$.

- ② is easy – just ask what $g'(7)$ is.
- For ①, you first ask: what is $g(7)$?

  You get the answer $g(7) = u$.

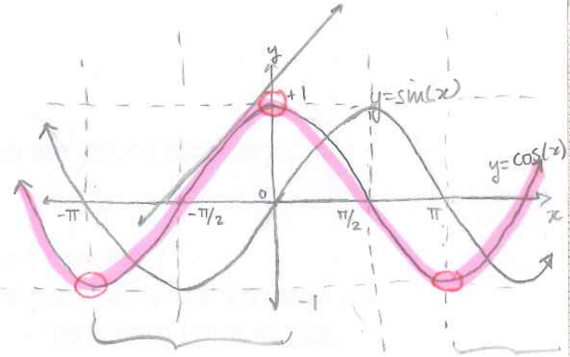
  Then ask: what is $\underline{f'(u)}$?

Final answer:

$$F'(7) = f'(u) \cdot g'(7).$$

3.7: Trig. Functions

Two big rules:



(1) $\frac{d}{dx}[\sin x] = \cos x$

(2) $\frac{d}{dx}[\cos x] = -\sin x$.

Proof of (1):

recall the def'n of the derivative: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. For us:

$$\frac{d}{dx}[\sin x] = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

Recall: $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$$\text{So, } \frac{d}{dx}[\sin x] = \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\cos x \left(\frac{\sin \Delta x}{\Delta x} \right) - \sin x \left(\frac{1 - \cos \Delta x}{\Delta x} \right) \right]$$ (just re-factoring)

$$= \cos x \underbrace{\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}} - \sin x \underbrace{\lim_{\Delta x \to 0} \left(\frac{1 - \cos \Delta x}{\Delta x} \right)}$$ (limit laws of sum & constant multiplication — note x is held constant by this limit in Δx)

Recall: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$.

$$\text{So, } \frac{d}{dx}[\sin x] = \cos x (1) - \sin x (0)$$

$$= \cos x.$$

NICE PROOF!

The proof for $\frac{d}{dx}[\cos x] = -\sin x$ is similar.

Ex. 1, p.170

$$\frac{d}{dx}\left[x^2 \sin x\right] = x^2 \frac{d}{dx}[\sin x] + \frac{d}{dx}[x^2]\cdot \sin x \quad \text{prod. rule}$$

$$= x^2 \cos x + 2x \sin x.$$

Ex. 2, p.171

$$\frac{d}{dx}\left[\frac{\cos x}{1-\sin x}\right] = \frac{(1-\sin x)\frac{d}{dx}[\cos x] - \cos x \frac{d}{dx}[1-\sin x]}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \quad \leftarrow \text{Recall: } \sin^2\theta + \cos^2\theta = 1$$

$$= \frac{\cancel{1-\sin x}}{(1-\sin x)^{\cancel{2}}}$$

$$= \frac{1}{1-\sin x}.$$

Ex. $y = (\cos x)^3$. Let $f(u) = u^3$, let $g(x) = \cos x$. Then $f'(u) = 3u^2$, and so $f'(g(x)) = 3(\cos x)^2$. Also, $g'(x) = -\sin x$, so by the chain rule,

$$\frac{dy}{dx} = f'(g(x))\,g'(x) = \underbrace{3(\cos x)^2}\,\underbrace{(-\sin x)} = -3\sin x(\cos x)^2.$$

Ex. $y = \cos(x^3)$. Let $f(u) = \cos u$, $g(x) = x^3$. Then $f'(u) = -\sin u$, so $f'(g(x)) = -\sin x^3$. Also, $g'(x) = 3x^2$. So by the chain rule,

$$\frac{dy}{dx} = f'(g(x))\,g'(x) = (-\sin x^3)(3x^2) = -3x^2 \sin x^3.$$

The other trig fns. . . .

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}.$$

Can use differentiation rules to find the derivatives of these, where the derivatives exist.

Ex. $\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos(x)\frac{d}{dx}[\sin x] - \sin(x)\frac{d}{dx}[\cos x]}{(\cos x)^2}$

(7)

$\frac{d}{dx}[\tan x] = \sec^2(x).$ $\quad = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{(\cos x)^2}$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$\boxed{* \text{ RECALL: } \sin^2\theta + \cos^2\theta = 1.}$

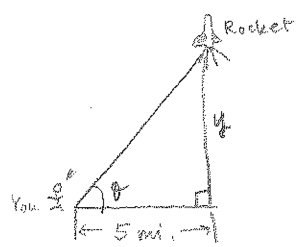
See p. 172 of the text for the rest of the formulas!

(8) $\frac{d}{dx}[\cot x] = -\csc^2(x)$

(9) $\frac{d}{dx}[\sec x] = \sec x \tan x$

(10) $\frac{d}{dx}[\csc x] = -\csc x \cot x.$

Ex. 12, p. 175



You're standing 5 mi. from the launchpad, and you observe the elevation θ of the line of sight to the rocket. At the time T when $\theta(T) = 60°$, θ is increasing by $3°$/sec. What is the velocity of the rocket at time T?

Given: $\theta(T) = 60° = \frac{\pi}{3}$ radians (use $\underline{180° = \pi \text{ rad}}$)

$$\left.\frac{d\theta}{dt}\right|_{t=T} = \theta'(T) = 3°/\text{sec} = \frac{3\,\cancel{\deg}}{\text{sec}}\cdot\underbrace{\frac{\pi \text{ rad}}{180\,\cancel{\deg}}}_{=1} = \frac{\pi}{60}\ \text{rad/sec}$$

Know: $\tan\theta = \frac{y}{5 \text{ miles}}$ $\left(\sin = \frac{O}{H},\ \cos = \frac{A}{H},\ \tan = \frac{O}{A}, \text{ where } \right)$

$$\Rightarrow y(\theta) = 5\tan\theta \Rightarrow y'(\theta) = 5\sec^2\theta$$

Want: $\left.\frac{dy}{dt}\right|_{t=T} = \left.\frac{dy}{d\theta}\frac{d\theta}{dt}\right|_{t=T} = \left.5\sec^2\theta\right|_{t=T}\cdot\left.\frac{d\theta}{dt}\right|_{t=T} = 5\sec^2\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{60} = \frac{\pi}{3}\ \frac{\text{miles}}{\text{sec}}.$

Recall: An exponential fn. is of the form $f(x) = a^x$, for $a > 0$ constant.

(Has a very different character than $f(x) = x^a$!)

Until now, we have been pretty cavalier abt. using exponential functions, but we haven't really defined them carefully.

Recall again: $a^0 = 1$ $\quad a^3 = a \cdot a \cdot a$

$a^1 = a$ $\quad a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$

$a^2 = a \cdot a$

$a^{-n} = \dfrac{1}{a^n}$, and $a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$

$a^{\sqrt{2}}, a^{\pi}, a^{e}$

So we know how to define the exponential for all rational exponents (that is, we can define $f(x) = a^x$ for $x \in \mathbb{Q}$).
$\longrightarrow \mathbb{Q} = \{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \}$.

What abt. irrational exponents?

We define: $a^x := \lim\limits_{r \to x} a^r$, with r rational.

Note that this is our definition — we chose it to be true — and it's no accident that this is almost what we need in order to make $f(x) = a^x$ continuous on all $\mathbb{R}$.

You will take it as given, for now, that the familiar rules of exponentiation work for irrational exponents, too!

- $a^{x+y} = a^x a^y$
- $(a^x)^y = a^{xy} = (a^y)^x$
- $a^{-x} = \dfrac{1}{a^x}$
- $(ab)^x = a^x b^x$.

(Can rigorously prove in Calc II/III.)

Derivatives of exponential, logarithmic fns.

$$f(x) = a^x \Rightarrow f'(x) = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^x(a^{\Delta x} - 1)}{\Delta x} = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}.$$

Looking at the graph, we see no reason why $f(x) = a^x$ should not be diff'ble — so we know that the limit $\lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$ should exist — but what is it?

It's obviously not a fn. of Δx, or of x — so it must just be a fn. of a. For now, let's call it $m(a)$.

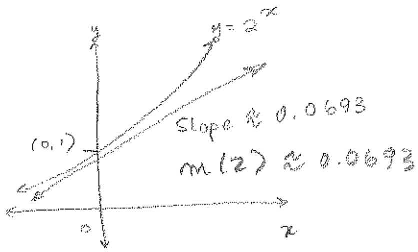
So, we get $\frac{d}{dx}\left[a^x\right] = m(a) \cdot a^x$, a CONSTANT MULTIPLE OF a^x ITSELF.

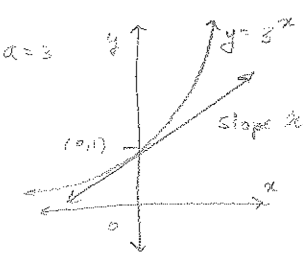
Recall: $f'(x)$ is the slope of the line tangent to the graph of f at x.

So for any a, $f'(0) = m(a) \cdot a^0 = m(a)$ — thus, $m(a)$ = slope of line tan. to $f(x) = a^x$ at the point $(0,1)$.

Check this out for a couple of examples:

$a = 2$





$m(3) \approx 1.099$

See above — these slopes are just $\lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$, so just compute $\frac{a^{\Delta x} - 1}{\Delta x}$ for successively smaller values of Δx (i.e., $\Delta x = 0.1$, 0.01, 0.001, etc.) until some kind of pattern is evident — this gives a good guess as to the limit.

Note that if $m(a)$ is a continuous fn. of a, then we can apply the IVT to show that for some number e in $(2,3)$, we have $m(e) = 1$, i.e.,

$$\boxed{\frac{d}{dx}\left[e^x\right] = e^x.}$$

(This is a useful property for a fn. to have!)

Ex. $\frac{d}{dx}\left[e^{kx}\right] = \frac{d}{dx}[kx]\, e^{kx} = ke^{kx}$, any const. k

Ex. $\frac{d}{dx}\left[x^3 e^{-x}\right] = 3x^2 e^{-x} + x^3(-e^{-x}) = x^2 e^{-x}(3-x)$.

Ex. $\frac{d}{dx}\left[e^{(x^2)}\right] = \frac{d}{dx}[x^2]\, e^{(x^2)} = 2x e^{(x^2)}$.

We will not worry abt. $a \neq e$ for another few minutes. For now, though...

Inverse Functions.

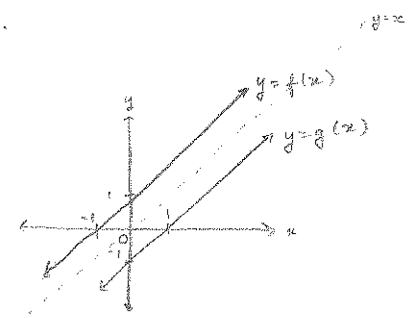
Def. $f(x)$ and $g(x)$ are said t/b inverse functions if both:

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x.$$
$$f \circ g = x \qquad\qquad g \circ f = x$$

ex. $f(x) = x+1$, $g(x) = x-1$.

$f(g(x)) = (x-1)+1 = x$

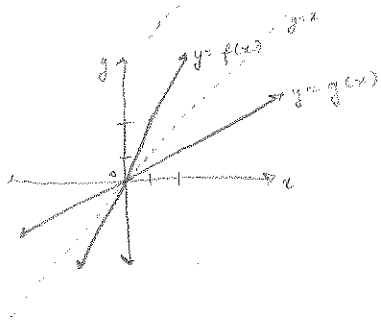
$g(f(x)) = (x+1)-1 = x$



ex. $f(x) = 2x$, $g(x) = \frac{x}{2}$

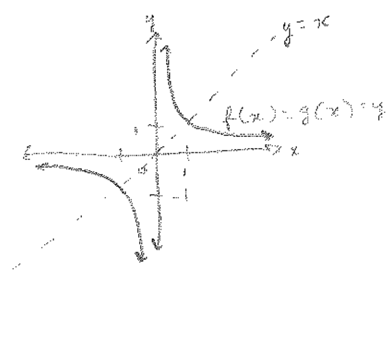
$f(g(x)) = 2\left(\frac{x}{2}\right) = x$;

$g(f(x)) = \frac{2x}{2} = x$.



ex. $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x}$

$f(g(x)) = g(f(x)) = \frac{1}{\left(\frac{1}{x}\right)} = x$



Inverse fns.

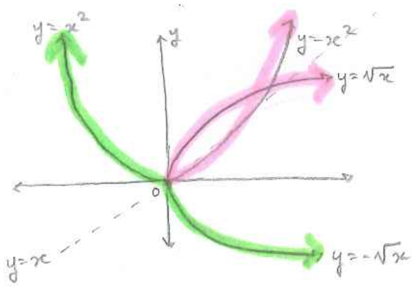
Sometimes have to watch out for the domain & range... $f: D \to R$

ex. $f(x) = x^2$, $g(x) = \sqrt{x}$.

$f: (-\infty, +\infty) \longmapsto [0, +\infty)$

$g: [0, +\infty) \longmapsto [0, +\infty)$.

Restrict the domain of f to $f: [0, +\infty) \to [0, +\infty)$ and the inverses work:



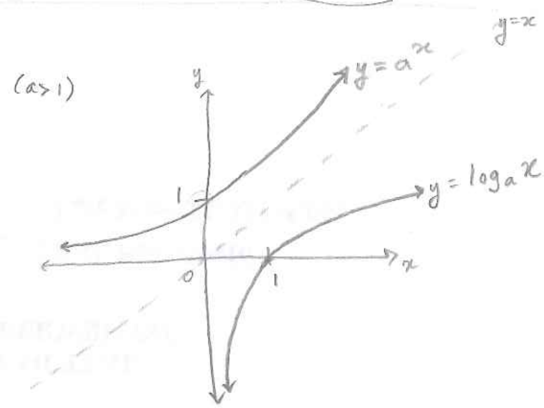
Or, consider $f(x) = x^2$, $f: (-\infty, 0] \to [0, +\infty)$

and $g(x) = -\sqrt{x}$, $g: (0, +\infty] \to (-\infty, 0]$

$$\boxed{\begin{array}{l} \text{So DOMAIN } f = \text{RANGE } g \\ \text{DOMAIN } g = \text{RANGE } f. \end{array}}$$

The big example of inverse fns.

$f(x) = a^x$, $g(x) = \log_a x$.



Recall: $\log_a y = x$ iff $a^x = y$.

So, $f(g(x)) = a^{\log_a x} = x$

$g(f(x)) = \log_a (a^x) = x$.

This is the way we defined the logarithm!

Note, $f: (-\infty, +\infty) \mapsto (0, +\infty)$

$g: (0, +\infty) \mapsto (-\infty, +\infty)$.

Natural logarithm

It's just $\log_e (x)$

$\ln (x)$

$\ln (y) = x$ iff $e^x = y$

Derivatives of Inverse Fns.

THEOREM. Let $I \subseteq \mathbb{R}$ (subset) be an open interval, let f be a diff'ble fn. defined on I with $f'(x) > 0$ for all $x \in I$, or $f'(x) < 0$ for all $x \in I$.

p.187

Then f has an inverse fn. g, g is diff'ble, and for all x in the domain of g,

$$g'(x) = \frac{1}{f'(g(x))}.$$

N.B.: We don't have the tools to prove this thm (wait until adv. calculus!)

N.B.: Important distinction between the following:

$$\text{I} \quad \forall x \in I, \quad f'(x) > 0 \text{ or } f'(x) < 0$$

and

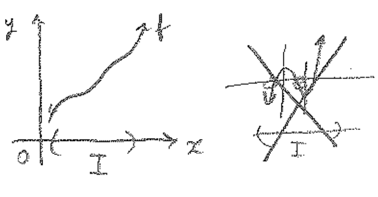
$$\longrightarrow \text{II} \quad (\forall x \in I \ f'(x) > 0) \text{ or } (\forall x \in I \ f'(x) < 0.)$$

Think: I "Each person in the world is either left-handed or right-handed".

II "Either each person in the world is left-handed, or each person in the world is right-handed".

N.B.: The fn. f needs to be "one-to-one" / "injective" on I — so, only one input for each output — in other words, in addition to the vertical line test, f should pass the horizontal line test.

$f'(x) < 0 \ \forall x \in I$, or $f'(x) > 0 \ \forall x \in I$ guarantees this — either the fn. is always increasing, or it is always decreasing.



Inverse fns & logarithms.

So, back to $\frac{d}{dx}\left[a^x\right] = m(a)\,a^x$.

We know that $f(x) = a^x$ and $g(x) = \log_a(x)$ are inverses, and we know that the inverse of a function is unique.

[Aside: why is a function's inverse unique?

Suppose f and g_1 are inverses, and f and g_2 are inverses.

Then $f(g_1(x)) = x$, $f(g_2(x)) = x$, $g_1(f(x)) = x$, $g_2(f(x)) = x$.

In particular, $g_2(x) = g_2\left(f[g_1(x)]\right) = g_1(x)$.

because $f(g_1(x)) = x$; because $g_2(f(a)) = a$]

So we can apply Theorem 1 on any open interval $I \subseteq \mathbb{R}$ (because for $a>0$, $f'(a) > 0$ as well) to obtain:

$$g'(x) = \frac{1}{f'(g(x))}, \quad \text{with } m(a)\,a^x = f'(x)$$

$$= \frac{1}{m(a)\,\underbrace{a^{\log_a x}}_{=1}} = \boxed{\frac{1}{m(a)\,x}}$$

In particular, for the natural exponential, there is a corresponding natural log: $\ln(x) = \log_e(x)$, and recall that $m(e) = 1$.

So $\boxed{\frac{d}{dx}\left[e^x\right] = e^x \quad \text{and} \quad \frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}}$

Let's use this to figure out what $m(a)$ is, finally!

Observe: $a = e^{\ln(a)} \Rightarrow a^x = (e^{\ln(a)})^x = e^{x\ln(a)}$.

Know that $\frac{d}{dx}[a^x] = m(a)\,a^x$.

$a^x = e^{x\ln(a)}$

Calculate using chain rule:

$$\frac{d}{dx}\left[e^{x\ln(a)}\right] = \ln(a)\,e^{x\ln(a)} = \ln(a)\,a^x.$$

recall: we already calculated
$\frac{d}{dx}[e^{kx}] = ke^{kx}$
using chain rule

Thus, $m(a)\,a^x = \ln(a)\,a^x \Rightarrow \boxed{m(a) = \ln(a).}$

So, finally, we have our rules for exponentials & logarithms:

$$\boxed{\frac{d}{dx}[a^x] = \ln(a)\,a^x}$$

AND

$$\boxed{\frac{d}{dx}[\log_a(x)] = \frac{1}{\ln(a)\,x}.}$$