Lecture 5: June 04, 2014.

TODAY:
- Logarithmic differentiation (end of 3.3)
- Derivatives of Algebraic fns. (3.4)
- Maxima/minima & applied optimization (3.5, 3.6)

Logarithmic Differentiation.

Recall: (1) $\log_a(xy) = \log_a(x) + \log_a(y)$

(2) $\log_a(x^y) = y\log_a(x)$

* Consequence: (3) $\log_a\left(\frac{1}{x}\right) = \log_a(x^{-1}) = -\log_a(x)$

* Consequence: (4) $\log_a\left(\frac{x}{y}\right) = \log_a(xy^{-1})$

$$= \log_a(x) + \log_a(y^{-1})$$

$$= \log_a(x) - \log_a(y)$$

(5) $\log_e(x) =: \ln(x)$

(6) $\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$

* Consequence: (7) $\frac{d}{dx}[\ln(x)] = \frac{d}{dx}[\log_e(x)]$

$$= \frac{1}{x\ln(e)}$$

$$= \frac{1}{x}$$

* Consequence: (8) $\frac{d}{dx}[\log_a(f(x))] = \frac{f'(x)}{f(x)\ln(a)}$

(chain rule)

These rules are very handy ...

ex. 12, p. 190

$$\frac{d}{dx}\left[\ln\sqrt{\frac{2x+3}{4x+5}}\right] = \frac{\frac{d}{dx}\left[\sqrt{\frac{2x+3}{4x+5}}\right]}{\sqrt{\frac{2x+3}{4x+5}}}$$ Rule (8)

$$= \frac{\frac{1}{2}\left(\frac{2x+3}{4x+5}\right)^{-1/2}\frac{d}{dx}\left(\frac{2x+3}{4x+5}\right)}{\sqrt{\frac{2x+3}{4x+5}}}$$ Chain rule

$$= \frac{\frac{d}{dx}\left[\frac{2x+3}{4x+5}\right](4x+5)}{2(2x+3)}$$ Arithmetic

Quotient rule

$$= \frac{(4x+5)\frac{d}{dx}(2x+3)-(2x+3)\frac{d}{dx}(4x+5)}{(4x+5)^2}\ \frac{4x+5}{2(2x+3)}$$

$$= \frac{2(4x+5)-4(2x+3)}{2(2x+3)(4x+5)}$$ Differentiating the polynomials

$$= \frac{4x+5-4x-6}{(2x+3)(4x+5)}$$

$$= -\frac{1}{(2x+3)(4x+5)}.$$

On the other hand...

$\ln(x^y) = y\ln(x)$

Rule (2)

$$\frac{d}{dx}\left[\ln\sqrt{\frac{2x+3}{4x+5}}\right] = \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{2x+3}{4x+5}\right)\right]$$

$$= \frac{d}{dx}\left[\frac{1}{2}\left(\ln(2x+3)-\ln(4x+5)\right)\right]$$ Rule (4)

$$= \frac{1}{2}\ \frac{\frac{d}{dx}(2x+3)}{2x+3} - \frac{1}{2}\ \frac{\frac{d}{dx}(4x+5)}{4x+5}$$ Rule (8)

$$= \frac{2}{2(2x+3)} - \frac{4}{2(4x+5)}$$ Polynomial derivs

$$= \frac{2(4x+5)-4(2x+3)}{2(2x+3)(4x+5)}$$ ← same as before, so get same result.

Using the log rules really made life simpler!

Logarithmic differentiation exploits this idea!

STEPS: ① Given $y = f(x)$

② Take natural log of both sides: $\ln y = \ln f(x)$

— SIMPLIFY —

Careful if f has negative values!

③ Differentiate: $\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\ln f(x)\right]$

④ Multiply by y: $\frac{dy}{dx} = y\frac{d}{dx}\left[\ln f(x)\right]$

$$= f(x)\frac{d}{dx}\left[\ln f(x)\right]$$

This might look like a more complicated process for now, but...

Ex. 14
p. 191

Find $\frac{dy}{dx}$, given $\boxed{y = x^{x-1}}$; $x > 0$.

NOTE: We can't do this using the rules we have now!

STEP 2 $\quad y = x^{x-1} \Rightarrow \ln(y) = \ln(x^{x-1})$
$$= (x-1)\ln(x)$$

$\ln(x^y) = y\ln(x)$ Rule ②

STEP 3
$$\Rightarrow \frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[(x-1)\ln(x)\right]$$

$$\Rightarrow \underbrace{\frac{1}{y}\frac{dy}{dx}}_{\text{Rule } ⑧} = \underbrace{(x-1)\frac{d}{dx}\left[\ln(x)\right] + \frac{d}{dx}\left[x-1\right](\ln(x))}_{\text{Prod. rule}}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x-1}{x} + \ln(x) = \frac{x-1+x\ln x}{x}$$

$$= \frac{x(1+\ln x)-1}{x}$$

STEP 4
$$\Rightarrow \frac{dy}{dx} = y\left(\frac{x(1+\ln x)-1}{x}\right) = \frac{x^{x-1}}{x}\left(x(1+\ln x)-1\right)$$

$$\boxed{= x^{x-2}\left[x(1+\ln x)-1\right]}$$

Section 3.4: Algebraic fns.

Recall Power rule: $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$, $n \in \mathbb{N} = \{1, 2, 3, \dots\}$.

We've been a little bold by using this rule not only for $n \in \mathbb{N}$, but for $n \in \mathbb{Z}$ and even $n \in \mathbb{Q}$.

Ex. $\frac{d}{dx}\left[x^{-2}\right] = -2x^{-3}$, $\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{1/2}\right] = \frac{1}{2}x^{-1/2}$

$= \frac{1}{2\sqrt{x}}$.

note, $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

The Generalized power rule. If $r \in \mathbb{Q}$, then

Thm. 1, p. 140

$$\frac{d}{dx}\left[\left(f(x)\right)^r\right] = r\left[f(x)\right]^{r-1} f'(x),$$

for those x where the RHS is defined.

Ex. 2, p. 140

$$\frac{d}{dx}\left[\sqrt{4-x^2}\right] = \frac{d}{dx}\left[(4-x^2)^{1/2}\right]$$

$$= \frac{1}{2}(4-x^2)^{-1/2}\frac{d}{dx}\left[4-x^2\right] \quad \text{G.P.L.}$$

$$= \frac{1}{2}(4-x^2)^{-1/2}(-2x) \quad \text{polynomial}$$

$$= \frac{-2x}{2\sqrt{4-x^2}}$$

$$= \frac{-x}{\sqrt{4-x^2}}.$$

This value does not exist when $4 - x^2 \le 0$,
i.e., when $x^2 \ge 4$.

So the derivative is defined only for $x \in (-2, 2)$

OPEN INTERVAL

So, we just saw a function whose derivative did not exist outside $(-2,2)$.

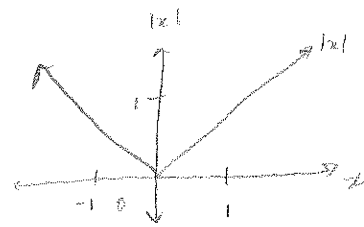
This is what makes algebraic fns. different from polynomial and rational fns. — the algebraic fns. might still be continuous at points where the derivatives don't exist.

[N.B. — The converse is actually false — we'll shortly have a theorem stating that differentiability $\Rightarrow$ continuity.]

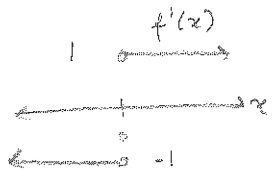
$\longrightarrow$ Ex. $f(x) = |x| = \sqrt{x^2}$ is an algebraic fn.

$$f'(x) = \frac{1}{2}(x^2)^{-1/2}\frac{d}{dx}(x^2)$$

G.P.L.



$$= \frac{2x}{2\sqrt{x^2}} = \frac{x}{\sqrt{x^2}}$$



$$= \frac{x}{|x|} = \begin{cases} -1, & x<0 \\ +1, & x>0 \end{cases}.$$

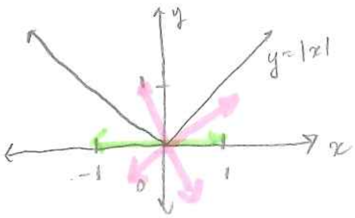
f is diff'ble everywhere, except POSSIBLY at $x=0$.

[N.B. — If the RHS of the GPL does not exist, note that the GPL doesn't guarantee the function is diff'ble — and this is different from guaranteeing the fn. is not diff'ble!

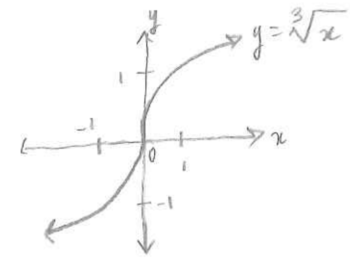
See: If an animal is a beagle, then it is a dog. I give you an animal that's not a beagle; then this statement can't guarantee I've given you a dog — but it also can't guarantee I've not given you a dog (since I might have given you a dalmatian).

This might remind you of the inverse fn. theorem from Monday?]

The trouble with $|x|$ at $x=0$ is the sharp angle!
$= \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$



... But sharp ∡'s aren't the only place we have trouble.

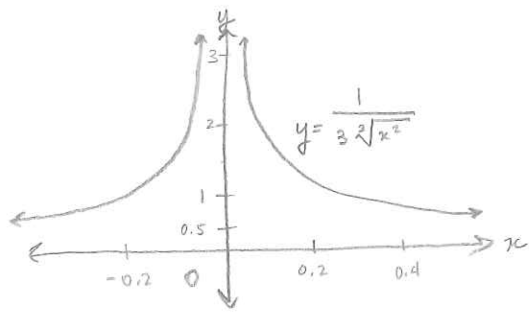


ex. 8, p. 142

$$y = \sqrt[3]{x} = x^{1/3}.$$

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}.$$

Let's look at the behavior of $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$ around $x=0$:



See that the derivative has no value at $x=0$.

Looking at the graph of f, the tan. line at $x=0$ is just a vertical line.

Recall: slope $= \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$.

For a vertical line, though, all points on the line have the same x-value — so

$$\text{slope} = \frac{\Delta y}{0} \quad ??$$

We saw even back in middle school that the slope of a vertical line is not defined.

Let's make a connection to tan. lines and fns. like $x^{1/3}$.

DEFINITION. $y = f(x)$ has a <u>vertical tangent line</u> at the point $(a, f(a))$ provided that f is cts. at a, and:

$$|f'(x)| \xrightarrow[x \to a]{} +\infty.$$

$\boxed{\text{Example } 9, \text{ p. } 142}$ Find all points on the curve

$$y = f(x) := x\sqrt{1-x^2}, \quad -1 \le x \le 1,$$

where the tangent line is either vertical or horizontal.

[N.B. — Such points are called <u>critical points</u>, and indeed, they are very important!]

(The tangent line is horizontal when $f'(x) = 0$
vertical when $|f'(x)| \longrightarrow +\infty$.

So we need to calculate $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[x\sqrt{1-x^2}\right] = x\frac{d}{dx}\left[\sqrt{1-x^2}\right] + \sqrt{1-x^2}\,\frac{d}{dx}[x] \qquad \text{PROD.}$$

$$= \frac{x\frac{d}{dx}[1-x^2]}{2\sqrt{1-x^2}} + \sqrt{1-x^2} \qquad \text{GPL}$$

$$= \frac{x(-2x)}{2\sqrt{1-x^2}} + \sqrt{1-x^2}$$

$$= \frac{-x^2+1-x^2}{\sqrt{1-x^2}} = \boxed{\frac{1-2x^2}{\sqrt{1-x^2}}.}$$

$f'(x) = 0$ when $1-2x^2 = 0$, i.e., when $x = \pm\sqrt{1/2}$.

We were asked for points on the curve, so we calculate $f(\pm\sqrt{1/2})$ to find the two points $(-\sqrt{\tfrac{1}{2}}, -\tfrac{1}{2})$ and $(\sqrt{\tfrac{1}{2}}, \tfrac{1}{2})$.

$\longrightarrow$ CONTINUED $\longrightarrow$

So, $f'(x) = \dfrac{1-2x^2}{\sqrt{1-x^2}}$.

Let's see where $f'(x)$ is undefined — that's where either

① the denominator is zero

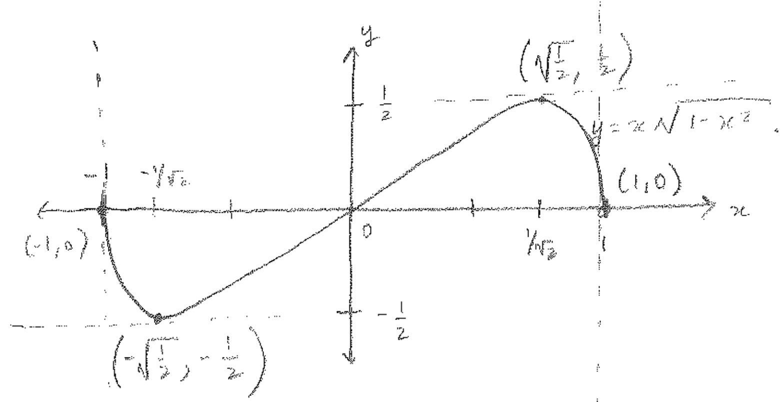
or

② the argument of the radical is negative.

For ①, this happens when $1-x^2=0$, i.e., when $x=\pm 1$.

For ②, this happens when $1-x^2<0$, i.e., when $|x|>1$. But those points were excluded from the domain by the problem statement.

So we need to look only at $x=\pm 1$. Note, $f(1)=f(-1)=0$, so the points where the tangent line is vertical are $(-1,0)$ and $(1,0)$.

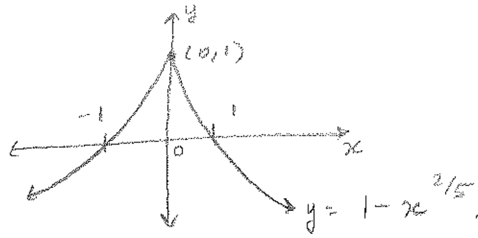


Ex. 10 p. 142

$f(x) = 1 - \sqrt[5]{x^2}$

$f'(x) = -\frac{2}{5} x^{-3/5}$

$= \dfrac{-2}{5\sqrt[5]{x^3}}$

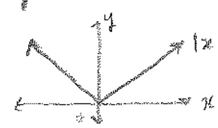


Note, $|f'(x)| \xrightarrow[x \to 0]{} +\infty$, and since f is cts. at $x=0$,

we have, by def'n, a vertical tan. line at $x=0$.

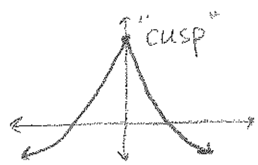
So, we saw a few examples of non-diff'ble, but cts., fns:

1. corner point



fn. is not diff'ble at $x=0$ because $\lim\limits_{x \to 0} f'(x)$ DNE

2. vertical tan. line



fn. is not diff'ble at $x=0$ because $\lim\limits_{x \to 0} |f(x)| = +\infty$.

So continuity $\not\Rightarrow$ diff'bility. But, actually:

diff'bility $\Rightarrow$ continuity:

THEOREM 2 p. 143 — If f is defined in a nbd. (neighborhood) of a, and if f is diff'ble at a, then f is cts. at a.

Proof: $\lim\limits_{x \to a} [f(x) - f(a)] = \lim\limits_{x \to a} \dfrac{x-a}{x-a}(f(x) - f(a))$

$= \left[\lim\limits_{x \to a} (x-a)\right]\left[\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x-a}\right]$

PRODUCT LAW — can use only because we know right-hand limit exists — this is where diff'bility comes in!

$=0$

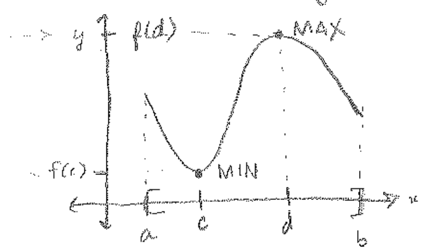
$= 0 \cdot f'(a) = 0 \Rightarrow \lim\limits_{x \to a} f(x) = f(a)$.

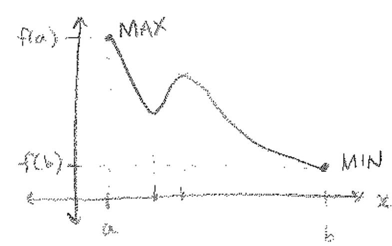
3.5 : Maxima & minima

Def. For f a function, $[a,b]$ a closed interval — we say $f(c)$ is the <u>minimum value</u> of f over $[a,b]$ and $f(d)$ is the <u>maximum value</u> of f over $[a,b]$ if

$$\forall x \in [a,b], \qquad f(c) \le f(x) \le f(d).$$

It's just what you would expect!

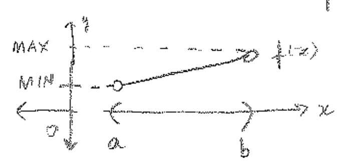




$(a = d, \ b = c)$

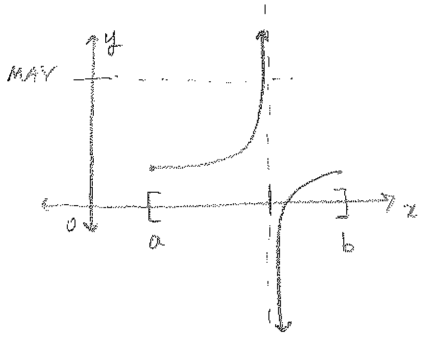
THEOREM 1, p. 147	A cts. fn. on a closed interval assumes (takes on) minimum and maximum values.

Closed intervals are important!



- f is cts., but the max & min are at the endpoints and aren't in (a,b)

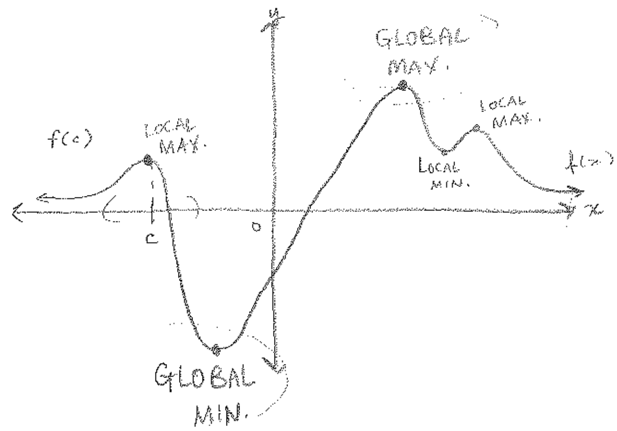
Continuity is important!



- $[a,b]$ is closed, but f has no max or min. at all!

Finding the extrema.) — extrema are max. or min. values

First - the difference btwn. local and global extrema:

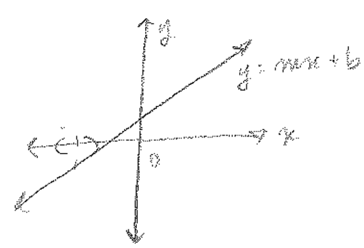


- f has a LOCAL $\begin{Bmatrix} \text{min.} \\ \text{max.} \end{Bmatrix}$ at $f(c)$ if for all x in some $\underbrace{\text{open interval containing}}_{\text{"neighborhood of" ("nbd. of")}}$ c, $f(x) \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} f(c)$.

- f has a GLOBAL $\begin{Bmatrix} \text{min.} \\ \text{max.} \end{Bmatrix}$ at $f(c)$ if for all $x \in \mathbb{R}$, $f(x) \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} f(c)$.

Some functions have neither local nor global extrema.

$y = mx + b$



We already had an intuitive guess about where we might find local extrema — recall HW 1.

Let's make this rigorous — THIS IS THE BIG IDEA FOR TODAY —

THEOREM 2 p. 148	Suppose f is diff'ble at c and is defined in a nbd. of c. If $f(c)$ is a local extremum of f, then $f'(c) = 0$.

(Proof in text.)

<u>N.B.</u> : This is a statement of the form $a \Rightarrow b$.
(i.e., $f(c)$ is a local extremum $\Rightarrow$ $f'(c) = 0$.)

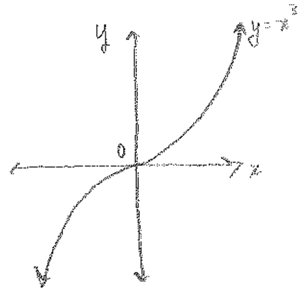
Recall that if $a \Rightarrow b$ is true, then the contrapositive NOT $b \Rightarrow$ NOT a is true as well.
(For us, this means $f'(c) \neq 0 \Rightarrow f(c)$ is NOT a local extremum.)

Also recall — the converse $b \Rightarrow a$ is <u>not</u> necessarily true!

In particular, the converse of Thm. 2 is untrue.

A proof that the converse is not true would consist of exhibiting a case where b holds but a does not — for us, this means we need to show a function and a value $f(c)$ such that $f'(c) = 0$, but $f(c)$ is not a local extremum.

Here it is: $f(x) = x^3$, $c = f(c) = 0$.



$f'(x) = 3x^2 \Rightarrow f'(0) = 0$, but check out the graph — obviously, $f(0) = 0$ is not a local extremum.

So, we say that $f'(c)=0$ is a necessary condition for $f(c)$ to be a local extremum — but it is NOT a sufficient condition.

IN PRACTICE...

Def. The number c in the domain of f is called a CRITICAL POINT of f if either:

① $f'(c)=0$

or

② $f'(c)$ does not exist.

THEOREM. 3, p. 159

Suppose $f(c)$ is the absolute max. or min. of the cts. fn. f over the closed interval $[a,b]$. Then either: ① c is a critical point of f
② c is an endpoint: $c=a$ or $c=b$.

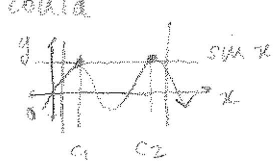
A procedure for finding minima & maxima

① Differentiate f and find its critical points — where $f'(c)=0$ or $f'(c)$ doesn't exist.

② Evaluate f at the critical pts. and the endpoints

③ The maximum is the largest among these fn. values; the minimum " " smallest " " .

(Note: the min/max. might not be unique — could have $f(c_1)=f(c_2)$ be extrema of f:



$c_1=\pi/2$, $c_2=3\pi/2$

Ex. $f(x) = 2x^3 - 3x^2 - 12x + 15$ over $[0,3]$.

① $f'(x) = 6x^2 - 6x - 12$

$= 6(x^2 - x - 2) = 6(x-2)(x+1)$.

$f'(x) = 0$ when $x = 2$ or $x = -1$, and is never undefined.

So the critical points are $x = -1$ and $x = 2$.

② Only one of the critical points is in $[0,3]$, so we evaluate:

$\rightarrow f(0) = 15$

$\rightarrow f(2) = -5$

$\rightarrow f(3) = 6$.

③ The max. value is $f(0) = 15$; the min. is $f(2) = -5$.

$(0,15)$ $(2,5)$

Ex. $f(x) = 2x^3 - 3x^2 - 12x + 15$ over $[-2,3]$.

① is the same.

② $f(-2) = 11$

$f(-1) = 22 \leftarrow$ max $(-1, 22)$

$f(2) = -5 \leftarrow$ min $(-5, 2)$

$f(3) = 6$

A similar procedure works for finding *global* extrema.

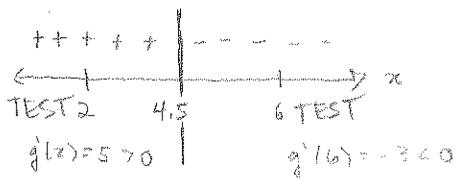
$\boxed{\text{Ex.}}$ $g(x) = 9x - x^2 - 1$. Domain $= \mathbb{R}$.

$g'(x) = 9 - 2x$, so $g'(x) = 0$ when $x = 4.5$.

So $x = 4.5$ is the only critical point of g.

Is it a max, a min, or neither?

N.B. — We can't compare function values — so it's easiest to inspect the derivative:



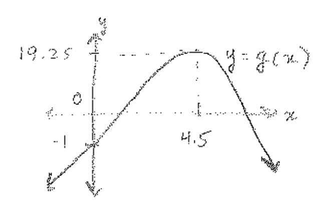
} Can choose any test points, as long as you have one in each interval btwn. critical points.

See — the derivative is positive for $x < 4.5$ and negative for $x > 4.5$, so the function has a local max, at $x = 4.5$ (the max. value is $g(4.5) = 19.25$).

Is the local max. a global one?

For this case, yes — see that $\lim_{x \to -\infty} g(x) = -\infty = \lim_{x \to \infty} g(x)$

The graph:



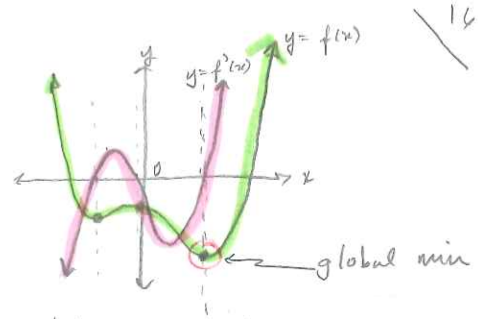
The global max is at $(4.5, 19.25)$.
The fn. has no global min.

Using the derivative to find extrema.

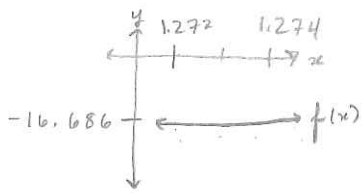
$\boxed{\text{ex } 7 \quad \text{p. } 152}$

$f(x) = 4x^4 - 11x^2 - 5x - 3$

$f'(x) = 16x^3 - 22x - 5$



Say we want to find the x-value of the global minimum of f. Zooming in a lot on f doesn't really help:



because f looks just like its (horizontal) tangent line when you get really, really close.

But zooming in on $f'(x)$: makes it clear that the critical point corresponding to the minimum is $x \approx 1.273$.