Lecture 7: June 16, 2014.

TODAY: • 3.9: Implicit differentiation, Related rates.
• 3.10: Newton's method
• Return midterms, discuss results...

3.9: Implicit differentiation, related rates.

So far, we have seen many examples of formulas that define a dependent variable explicitly as a fn. of the independent variable:

$$y = x^3 \sin x$$

$$y = x^2 + 7$$

$$y = e^x$$

etc.

We can also show the relationship between the dependent and independent variables implicitly by giving an equation that isn't "already" solved for the independent variable!

① $x^2 + y^2 = 100$

② $x - y^2 = 0$

etc.

These two examples are particularly convenient to solve for y:

① $x^2 + y^2 = 100 \iff y^2 = 100 - x^2$

$\iff y = \sqrt{100 - x^2}$ or $y = -\sqrt{100 - x^2}$.

② $x - y^2 = 0 \iff y^2 = x$

$\iff y = \sqrt{x}$ or $y = -\sqrt{x}$.

Each equation implicitly defines two functions.

Not all equations are so easy to solve, e.g.,

$$y^3 + x^3 = 3xy$$

$$\sin(x+2y) = 2x\cos y$$

etc.

If we want to find the derivative $\frac{dy}{dx}$, we should use

IMPLICIT DIFFERENTIATION.

Recall logarithmic differentiation, e.g.,

$$y = (2\sqrt{x})^{\sqrt{x}} \Leftrightarrow \ln(y) = \ln(2\sqrt{x})^{\sqrt{x}}$$
$$= \sqrt{x}\,\ln(2\sqrt{x})$$

So, $\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\sqrt{x}\,\ln(2\sqrt{x})\right]$

Chain rule $= \frac{1}{y}\frac{d}{dx}[y]$

$= \frac{1}{y}\frac{dy}{dx}$

Prod. rule $= \sqrt{x}\,\frac{d}{dx}\left[\ln(2\sqrt{x})\right] + \ln(2\sqrt{x})\,\frac{d}{dx}\left[2\sqrt{x}\right]$

Chain rule $= \sqrt{x}\,\dfrac{\frac{d}{dx}[2\sqrt{x}]}{2\sqrt{x}} + \ln(2\sqrt{x})\,\frac{d}{dx}\left[2\sqrt{x}\right]$

Power law $= \frac{2}{2\sqrt{x}}\left[\frac{1}{2} + \ln(2\sqrt{x})\right]$

Thus, $\frac{dy}{dx} = \frac{y}{2\sqrt{x}}\left[1 + 2\ln(2\sqrt{x})\right] = (2\sqrt{x})^{\sqrt{x}-1}\left[1 + 2\ln(2\sqrt{x})\right]$.

We had an IMPLICIT relationship between y and x — that is, $\ln(y) = \sqrt{x}\,\ln(2\sqrt{x})$, we differentiated both sides and — CRUCIALLY! — used the chain rule to obtain $\frac{dy}{dx}$, which we solved for algebraically.

Let's try a similar procedure, starting w/ a different implicit relationship:

e.g., $x - y^2 = 0 \Rightarrow \underbrace{\frac{d}{dx}\left[x - y^2\right]} = \underbrace{\frac{d}{dx}[0]}_{=0}$ (differentiate both sides)

$= \frac{d}{dx}[x] - \frac{d}{dx}[y^2]$

$= 1 - 2y\frac{dy}{dx}$.

Recall chain rule: If $y = y(x)$, then $\frac{d}{dx}[g(y)] = \frac{dg}{dy}\frac{dy}{dx}$.

In our case, $g(y) = y^2 \Rightarrow \frac{d}{dx}[y^2] = \frac{d}{dy}[y^2]\frac{dy}{dx}$

$= 2y\frac{dy}{dx}$.

Thus, $1 - 2y\frac{dy}{dx} = 0 \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{2y}.}$

Note: Can have y on RHS! (Couldn't with logarithmic diff.)

With this simple example, let's try it the "old" way - solving for y explicitly, then differentiating:

$x - y^2 = 0 \Rightarrow$ either ① $y = +\sqrt{x}$ or ② $y = -\sqrt{x}$.

Case ①: $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, Case ②: $\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$.

In both cases, see: $\frac{dy}{dx} = \frac{1}{2y}$.

(We won't always be able to check our work this way.)

So, STEPS: ① DIFFERENTIATE BOTH SIDES OF EQUATION THAT GIVES THE IMPLICIT RELATIONSHIP, USING THE CHAIN RULE FOR ALL INSTANCES OF "y".

② SOLVE ALGEBRAICALLY FOR "$\frac{dy}{dx}$".

Example. The "folium of Descartes": $x^3+y^3=3xy$.

$$\frac{d}{dx}\left[x^3+y^3\right] = \frac{d}{dx}\left[3xy\right]$$

$$\text{LHS} = \frac{d}{dx}\left[x^3\right] + \frac{d}{dx}\left[y^3\right] \qquad \text{sum rule}$$

$$= 3x^2 + 3y^2\frac{dy}{dx} \qquad \text{power law + chain rule}$$

$$\text{RHS} = \frac{d}{dx}\left[3x\right]y + 3x\frac{d}{dx}\left[y\right] \qquad \text{product rule}$$

$$= 3y + 3x\frac{dy}{dx}.$$

Thus,

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(y^2-x\right) = y - x^2 \quad \Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{y-x^2}{y^2-x}}.$$

Suppose we're given a point on the folium — $\left(\frac{3}{2}, \frac{3}{2}\right)$.

[Aside — check that $\underbrace{\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^3}_{} = 3\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)$.]

$$= 2\left(\frac{3}{2}\right)^3$$

$$= 3\left(\frac{3}{2}\right)^2 \quad ✓$$

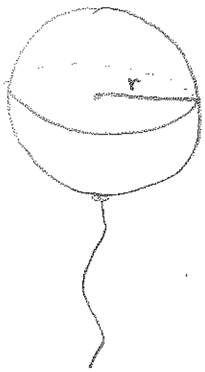
What is the slope of the line tangent to the folium at $\left(\frac{3}{2}, \frac{3}{2}\right)$? — Of course, it is $\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2},\frac{3}{2}\right)}$,

i.e., $$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2},\frac{3}{2}\right)} = \frac{\left(\frac{3}{2}\right)-\left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)} = -1.$$

So, the tan. line (point-slope): $y - \frac{3}{2} = -1\left(x-\frac{3}{2}\right)$.

RELATED-RATES problems involve two or more quantities varying with time, and some relationship — often implicit — between these quantities.

<u>Recall</u>: the "balloon" problem:



$V = \frac{4}{3}\pi r^3$

At time $t = T$ when $r(T) = 5$ cm, the radius r increases at $0.2 \ \frac{\text{cm}}{\text{sec}}$.

At what rate is the volume increasing at $t = T$?

This is a <u>related rates problem</u>, and the relationship btwn. r and V is EXPLICITLY defined: $V = \frac{4}{3}\pi r^3$.

This made the problem easy:

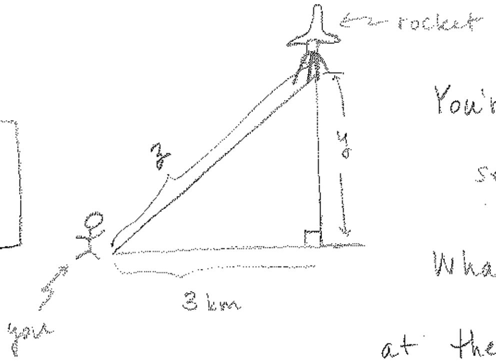
$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$= \frac{4}{3}\pi \frac{d}{dt}\left[r^3\right] = \frac{4}{3}\pi\,(3r^2)\,\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

so $$\left.\frac{dV}{dt}\right|_{t=T} = 4\pi (r(T))^2 \left.\frac{dr}{dt}\right|_{t=T} = 4\pi\,(5\text{ cm})^2\left(0.2\ \frac{\text{cm}}{\text{sec}}\right) = 20\pi\ \frac{\text{cm}^3}{\text{sec}}.$$

But, now that we have implicit differentiation in our toolbox, even <u>impli</u>citly defined relationships aren't hard to work with.

Example 6, p. 197



You're 3 km from the rocket's launchpad, seeing a rocket that is z km away. What is the vertical speed of the rocket at the instant $(t = T)$ when $z(T) = 5$ km and $\left.\frac{dz}{dt}\right|_{t=T} = 5000$ km/hr ?

See (Pythagorean thm.) that $y^2 + 3^2 = z^2$

$$\Rightarrow \quad y^2 + 9 = z^2 .$$

Want to find $\left.\frac{dy}{dt}\right|_{t=T}$, so differentiate (implicit):

$$\frac{d}{dt}\left[y^2 + 9\right] = \frac{d}{dt}\left[z^2\right]$$

LHS: $\frac{d}{dt}\left[y^2\right] + \frac{d}{dt}[9] = 2y\frac{dy}{dt}$

RHS: $\frac{d}{dt}\left[z^2\right] = 2z\frac{dz}{dt}$

So, $2y\frac{dy}{dt} = 2z\frac{dz}{dt}$. Solve algebraically:

$$\Rightarrow \quad \frac{dy}{dt} = \frac{z}{y}\frac{dz}{dt} .$$

Now, when $z(T) = 5$ km, have $y^2 + 9 = 5^2 \Rightarrow y(T) = 4$ km.

So, at $t = T$, have $\left.\frac{dy}{dt}\right|_{t=T} = \frac{z(T)}{y(T)}\left.\frac{dz}{dt}\right|_{t=T} = \frac{5\text{ km}}{4\text{ km}}\left(5{,}000\ \frac{\text{km}}{\text{hr}}\right)$

$$= \boxed{6{,}250\ \text{km/hr}} .$$

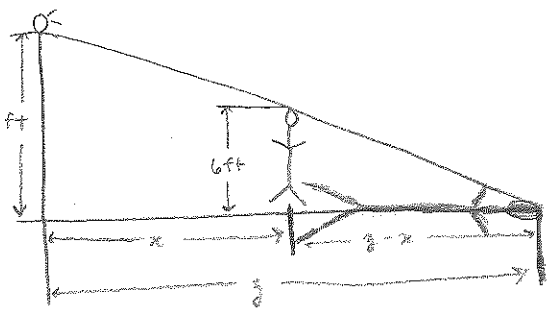
NOTES:
1. Use units as a reality check.
2. Substitute only after differentiating.
3. Know what the units are after differentiating.

STEPS:

1. Draw (& label!) a picture
2. Write down variables & known rates of change.
3. Write down an equation relating the important variables
4. Differentiate (implicitly?) $\underbrace{\text{w.r.t.}}_{\text{"with respect to"}}$ time t.
5. Substitute the known values into the resulting eq'n.

Example 7, p. 198

A man 6 ft. tall walks at 8 ft/sec away from a light on top of an 18 ft. pole. How fast is the tip of his shadow moving along the ground when he is 100 ft. from the pole?



Distance of man from pole: x

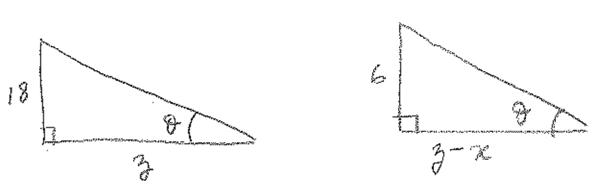
Distance of shadow tip from pole: z

—— " —————— man: $z - x$

Given that $\frac{dx}{dt} = 8\ \frac{\text{ft}}{\text{sec}}$ is constant.

Want to find the value of $\left.\frac{dz}{dt}\right|_{t=T}$, where $x(T) = 100$ ft.

Similar triangles:



Know $\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$

$= \frac{z}{18}$ on the one hand,

and $= \frac{z-x}{6}$ on the other.

So, $\frac{z}{18} = \frac{z-x}{6} \Rightarrow \boxed{2z = 3x.}$

Differentiate: $\frac{d}{dt}[2z] = \frac{d}{dt}[3x] \Rightarrow 2\frac{dz}{dt} = 3\frac{dx}{dt}$, so at

$t=T$, we have $\left.\frac{dz}{dt}\right|_{t=T} = \frac{3}{2}\left.\frac{dx}{dt}\right|_{t=T} = \frac{3}{2}\left(8\ \frac{\text{ft}}{\text{sec}}\right) = \boxed{12\ \text{ft/sec.}}$

3.10: Newton's method, Successive approximations

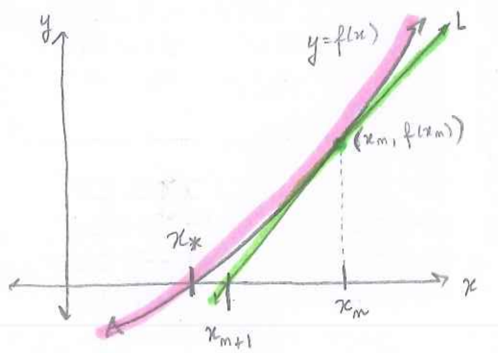
Let's say we want a numerical approximation of the solution to

$$f(x) = 0,$$

for some known function f, and moreover, we have a "sufficiently close" initial guess x_0.

Newton's method gives us a formula for producing a closer guess x_1; we can apply Newton's method again using x_1 as a guess, and thereby obtain an even closer guess x_2, and so on.

Here's how:



x_{m+1} should be the spot where L, the line tangent to the graph of $y = f(x)$ at $x = x_m$, intersects the x-axis.

In mathematics:

Find eq'n of L: know it intersects $(x_m, f(x_m))$ and has slope of $f'(x_m)$. Thus:

eq'n of L: $\quad y - f(x_m) = f'(x_m)(x - x_m).$

Solve this eq'n for x_{m+1} when $y = 0$:

$$-f(x_m) = f'(x_m)(x_{m+1} - x_m)$$

$$\Downarrow$$

$$\boxed{x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)}.}$$

"ITERATIVE FORMULA".

Example 2, p. 208

Find $\sqrt{2}$ accurate to 9 decimal places.

① INTERPRET PROBLEM AS SOLVING $f(x)=0$.

For us, $x=\sqrt{2}$ will solve $x^2-2=0$, so let $f(x)=x^2-2$.

② WRITE ITERATIVE EQ'N, SUBSTITUTING IN f AND f':

For us, $f'(x)=2x$, so the iterative eq'n:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2-2}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + 2}{2x_n}$$

$$= \frac{x_n^2+2}{2x_n}$$

③ CHOOSE AN INITIAL GUESS

Let's take $x_0 = 1$

④ ITERATE AS MANY TIMES AS NEEDED:

$$x_1 = \frac{x_0^2+2}{2x_0} = \frac{1^2+2}{2(1)} = \frac{3}{2} = 1.5$$

$$x_2 = \frac{x_1^2+2}{2x_1} = \frac{\left(\frac{3}{2}\right)^2+2}{2\left(\frac{3}{2}\right)} = \frac{17}{12} = 1.41\overline{6}$$

$$x_3 = \frac{x_2^2+2}{2x_2} = \frac{\left(\frac{17}{12}\right)^2+2}{2\left(\frac{17}{12}\right)} = \frac{3462}{2448} \approx 1.414215686$$

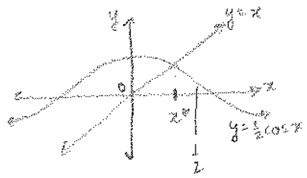
$$x_4 = \frac{x_3^2+2}{2x_3} = \frac{\left(\frac{3462}{2448}\right)^2+2}{2\left(\frac{3462}{2448}\right)} = \frac{665857}{470832} \approx \boxed{1.414213562}$$

$$x_5 = \frac{x_4^2+2}{2x_4} = \frac{\left(\frac{665857}{470832}\right)^2+2}{2\left(\frac{665857}{470832}\right)} = \dots \approx \boxed{1.414213562}$$

SEE: 1st 9 digits unchanged

Example. 4, p. 209

The eq'n $x = \frac{1}{2}\cos x$ has a sol'n near $x^* \approx 0.5$.
Find a 5-digit approximation to this solution.



① WRITE $f(x)$.

$f(x) = x - \frac{1}{2}\cos(x)$ could work, but

$f(x) = 2x - \cos(x)$ could be easier.

(See: $x^* - \frac{1}{2}\cos(x^*) = 0$ when $2x^* - \cos(x^*) = 0$, so the sol'n x^* is the same.)

② WRITE ITERATIVE FORMULA.

$f(x) = 2x - \cos(x)$ implies $f'(x) = 2 + \sin(x)$, so

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} = \boxed{x_n - \frac{2x_n - \cos x_n}{2 + \sin x_n}}.$$

③ CHOOSE AN INITIAL GUESS.

Pick $x_0 = 0.5$ (given.)

④ ITERATE AS NEEDED!

$$x_1 = x_0 - \frac{2x_0 - \cos x_0}{2 + \sin x_0} = 0.5 - \frac{2(0.5) - \cos(0.5)}{2 + \sin(0.5)} \approx 0.45063$$

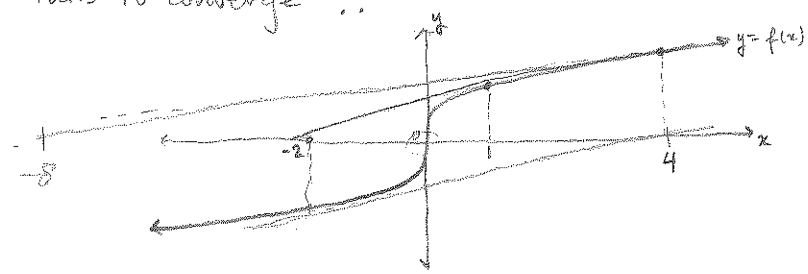
$$x_2 = x_1 - \frac{2x_1 - \cos x_1}{2 + \sin x_1} = 0.45063 - \frac{2(0.45063) - \cos(0.45063)}{2 + \sin(0.45063)} \approx \boxed{0.45018}$$

$$x_3 = x_2 - \frac{2x_2 - \cos x_2}{2 + \sin x_2} = 0.45018 - \frac{2(0.45018) - \cos(0.45018)}{2 + \sin(0.45018)} \approx \boxed{0.45018}$$

1st 5 digits the same.

Sometimes, Newton's method fails to converge !!

$\boxed{\text{Example } 6, \text{ p. } 210}$ $f(x) = x^{1/3}$



See: $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

so, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\left(\frac{1}{3x_n^{2/3}}\right)} = x_n - 3x_n = -2x_n$.

Start w/ a guess of $x_0 = 1$;

$x_1 = -2x_0 = -2$

$x_2 = -2x_1 = -2(-2) = 4$

$x_3 = -2x_2 = -2(4) = -8$

$\vdots$

These get <u>further away</u> from the actual root, $x^* = 0$.

We'd rather use some <u>other</u> method here (you are not responsible for knowing any other methods for now).

Cool stuff at the end of 3.10 that we won't cover in class — fractal-boundaried "basins of attraction" — check it out <u>if interested</u> (not on HW, WW, or exam).