Lecture 8: June 18, 2014.

ANNOUNCEMENT !!

NO IN-CLASS LECTURE MONDAY OR WEDNESDAY

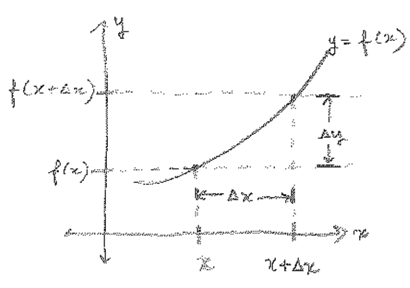
- TWO CLASS CAPTURES WILL BE ON MY WPI !
  (YOU MUST WATCH THEM)
- THERE WILL BE WEBWORK AND HOMEWORK !
- YOU CAN E-MAIL ME WITH QUESTIONS ALL NEXT WEEK.

TODAY: 4.2: Linear approximation, Differentials
4.3: Mean Value Theorem.

4.2: Increments.

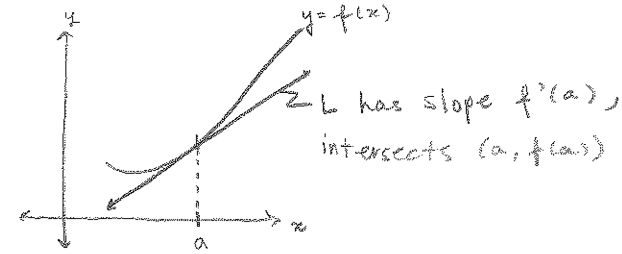
Recall: Δx – "Change in x" – "Increment in x"

$\Delta y = f(x+\Delta x) - f(x)$ "Change in y" – "Increment in y"



$\dfrac{\Delta y}{\Delta x}$ is the slope of the secant line connecting $(x, f(x))$ to $(x+\Delta x, f(x+\Delta x))$.

Also recall: $f'(a) = \left.\frac{dy}{dx}\right|_{x=a} := \left.\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right)\right|_{x=a}$ is the slope of the tangent line at $(a, f(a))$.



The equation of the line tangent to $f(x)$ at $x = a$:

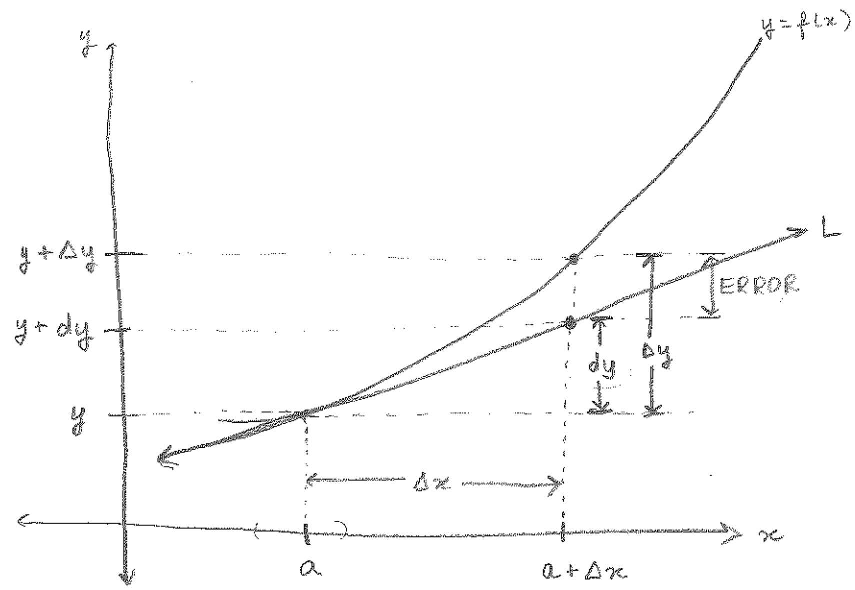
$$y - f(a) = f'(a)(x-a)$$

$(a, f(a))$
slope $f'(a)$

$$\boxed{\underset{L(x)}{\overset{\parallel}{y}} = f(a) + f'(a)(x-a).}$$

We call this the LINEAR APPROXIMATION TO $f(x)$ NEAR THE POINT $x = a$.

How good an approximation is it?



$$\left.dy\right|_{x=a} := L(a+\Delta x) - f(a)$$
$$= f'(a)\,\Delta x.$$

"differential"

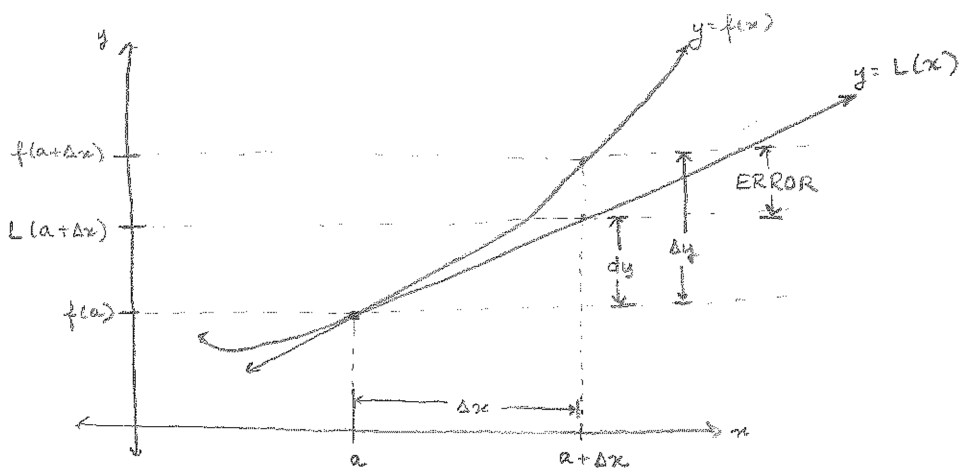
The *error* in a linear approximation:

In general,

$$\text{ERROR} = \text{ACTUAL VALUE} - \text{APPROXIMATE VALUE}.$$

So, the error in the linear approximation

$$\begin{aligned}\text{ERROR} &= f(x) - L(x) \\ &= f(x) - [f(a) + f'(a)(x-a)] \\ &= \underbrace{f(x) - f(a)}_{\Delta y} + \underbrace{f'(a)(x-a)}_{dy} \qquad (x = a + \Delta x) \\ &= \Delta y - dy.\end{aligned}$$



So, $\text{ERROR} = \Delta y - f'(a)\,\Delta x = \left[\dfrac{\Delta y}{\Delta x} - f'(a)\right]\Delta x$,

and note:

$$\begin{aligned}\lim_{\Delta x \to 0} \text{ERROR} &= \left[\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x} - f'(a)\right)\right]\left[\lim_{\Delta x \to 0} \Delta x\right] \\ &= \left[\left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\right) - f'(a)\right]\left[\lim_{\Delta x \to 0} \Delta x\right] \\ &= [f'(a) - f'(a)]\left[\lim_{\Delta x \to 0} \Delta x\right] \\ &= 0\end{aligned}$$

"The closer x is to a, the more accurate is $L(x)$ as a linear approx. to $f(x)$!"

So, Δy is the actual increment $f(x+\Delta x) - f(x)$,

and the differential $dy = f'(x)\,\Delta x$ is the approximation – the change in y that would occur if it were to change at the fixed rate $f'(x)$ over the interval $(x, x+\Delta x)$.

Example 1, p. 227 Find the linear approx to $f(x) = \sqrt{1+x}$ near $a = 0$.

Just means "write the equation of the tangent line".

Need: (1) a point

$$f(0) = \sqrt{1+0} = 1$$

our point is $(0, 1)$

(2) a slope

$$f'(x) = \frac{d}{dx}\left[\sqrt{1+x}\right] \overset{\frac{d}{dx}\left[(1+x)^{1/2}\right]}{=} \frac{1}{2\sqrt{1+x}} \frac{d}{dx}[1+x] = \frac{1}{2\sqrt{1+x}}$$

so

$$f'(0) = \frac{1}{2\sqrt{1+0}} = \frac{1}{2}.$$

The line:

$$y - 1 = \frac{1}{2}(x - 0) \Rightarrow y = \frac{1}{2}x + 1.$$

The approximation: $\sqrt{1+x} \approx \frac{1}{2}x + 1$ near $x = 0$.

$$\sqrt{1.001} \approx \frac{1}{2}(0.001) + 1 = 1.0005 \qquad 1.0005072145\ldots$$

NOTE abt. accuracy...

The linear approx. in example 1, $L(x) = \frac{1}{2}x + 1$, is closer to the function value $f(x) = \sqrt{1+x}$ when x is closer to 0.

See:

$$\left.\begin{aligned} f(0.1) &= \sqrt{1.1} \approx 1.04881 \\ L(0.1) &= \tfrac{1}{2}(0.1) + 1 = 1.05 \end{aligned}\right\} \text{ 2 decimal places of accuracy}$$

$$\left.\begin{aligned} f(0.03) &= \sqrt{1.03} \approx 1.014889 \\ L(0.03) &= \tfrac{1}{2}(0.03) + 1 = 1.015 \end{aligned}\right\} \text{ 3 decimal places}$$

But...

$$\left.\begin{aligned} f(2) &= \sqrt{3} \approx 1.73205\ldots \\ L(2) &= \tfrac{1}{2}(2) + 1 = 2 \end{aligned}\right\} \text{ very poor approximation}$$

Check the plot on Wolfram Alpha – graph $f(x)$ and $L(x)$ on the same axes, and zoom in on $x = 0$.

Useful for approximating roots in certain scenarios:

Example 2, p. 228 Use the lin. approx. formula to approximate $(122)^{2/3}$.

Note that $(125)^{2/3} = (\sqrt[3]{125})^2 = 5^2 = 25$.

* Crucial that 122 is somewhat close to 125, which has an exact value for $125^{2/3}$.

Let $\boxed{f(x) = x^{2/3}}$, and note that $\boxed{f(125) = 25}$.

We find the linear approx. to $f(x)$ around $\boxed{a = 125}$, and we plug in $x = 122$ to get an approximation of $(122)^{2/3}$

So, $f'(x) = \frac{2}{3} x^{-1/3}$, and $f'(125) = \frac{2}{3\sqrt[3]{125}} = \frac{2}{3(5)} = \frac{2}{15}$.

Thus, the local linear approximation to f close to 125:

$$L(x) - 25 = \frac{2}{15}(x - 125)$$

$$\Rightarrow \quad L(x) = \frac{2}{15}x + \frac{25}{3}$$

Thus, $L(122) = \frac{2}{15}(122) + \frac{25}{3} = \frac{123}{5} = 24.6$

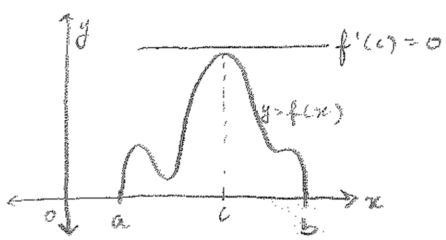
So, $\boxed{(122)^{2/3} \approx 24.6}$ (In reality, $(122)^{2/3} \approx 24.5984$)

We first start with a similar theorem:

ROLLE'S THEOREM (1690!)

HYP.
- Suppose f is cts. on $[a,b]$ and diff'ble on (a,b).
- If $f(a) = 0 = f(b)$,

Then $\underbrace{\exists}_{\text{"there exists"}} c \in (a,b)$ $\underbrace{\text{s.t.}}_{\text{"such that"}}$ $f'(c) = 0$. CONCLUSION



Proof. Because f was cts. on the closed interval $[a,b]$, it achieves its maximum and minimum values on $[a,b]$.

Case I. f has some positive values.
Then one of these positive values is the maximum - call it $f(c)$. We know $c \neq a$ and $c \neq b$, since $f(c) > 0$ but $f(a) = f(b) = 0$. Thus, $c \in (a,b)$, so f is differentiable at $x = c$. Its derivative there is thus $f'(c) = 0$.

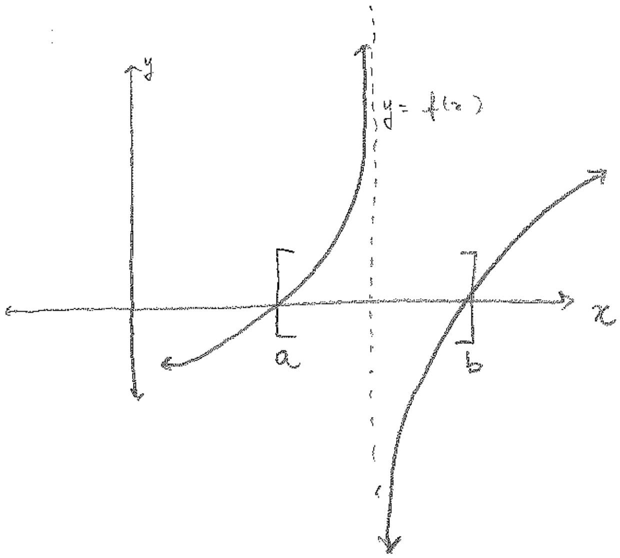
Case II. f has some negative values.
Similarly, we argue that one of these negative values is the minimum $f(c)$, and $c \in (a,b)$, so $f'(c)$ exists and is equal to zero.

Case III. f has no positive values and no negative values.
Then $f(x) \equiv 0$, so $f'(x) = 0$ everywhere in (a,b). □

So, we needed f to be cts. on $[a,b]$, diff'ble on (a,b), and $f(a) = 0 = f(b)$.

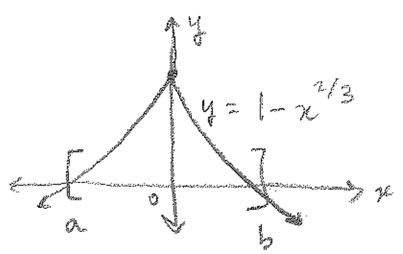
What if these don't hold?

① f not cts. on $[a,b]$:



has no horizontal tangent line on (a,b)

② f not diff'ble on (a,b):

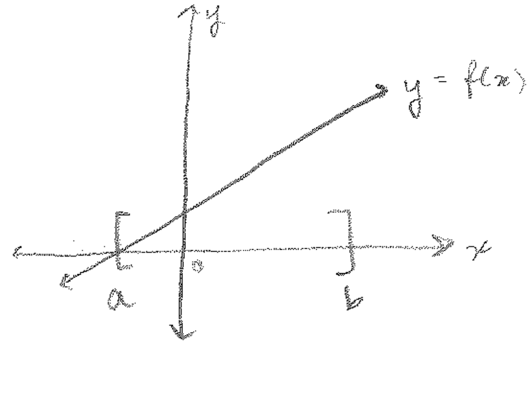


not diff'ble at $x = 0$.

has no horizontal tan. line in (a,b)

③ $f(a)$ or $f(b) \neq 0$

has no horizontal tangent line on (a,b)

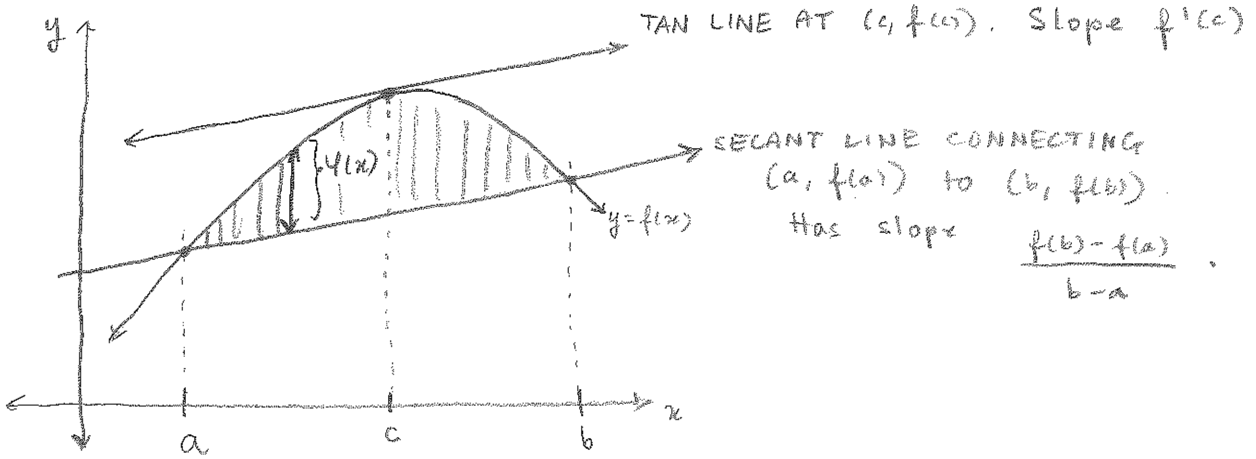


MEAN VALUE THEOREM.

Suppose f is cts. on $[a,b]$ and diff'ble on (a,b).

Then $\exists c \in (a,b)$ s.t.

$$\underbrace{f'(c)}_{\text{instantaneous rate of change of } f \text{ at } x=c} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{avg. rate of change of } f \text{ over } [a,b]}.$$



The instantaneous rate of change at $x=c$

is equal to

the average rate of change on the interval $[a,b]$.

Proof. Secant line has eq'n $y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$; let $\varphi(x)$ be the function $\varphi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$, so φ is cts. on $[a,b]$, diff'ble on (a,b), and $\varphi(a) = 0 = \varphi(b)$.

Apply Rolle's theorem — $\exists c \in (a,b)$ s.t. $\varphi'(c) = 0$, i.e., we have $\varphi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$, so finally,

$$f'(c) = \frac{f(b)-f(a)}{b-a}. \qquad \square$$

$\boxed{\text{Example } 4, \text{p.} 239}$ We drive from Kristiansand, Norway to Oslo, almost exactly 350 km, in 4 hours.

Let $f(t)$ denote the distance already travelled at time t, and assume f is differentiable (a strong assumption?).

Know $f(0) = 0$, $f(4) = 350$ $\Rightarrow$ avg speed $= \dfrac{350-0}{4-0} = \underline{87.5} \ \dfrac{\text{km}}{\text{hr}}$.

MVT $\Rightarrow$ $\exists\, c \in (0,4)$ s.t. $f'(c) = 87.5$ km/hr — that is, at time $t=c$, our <u>instantaneous</u> velocity is equal to our <u>average</u> velocity over the entire trip.

Recall: $f(x) = C$ constant $\Rightarrow$ $f'(x) = 0$.

We now use the MVT to prove the converse, i.e.,

$$f'(x) = 0 \quad \Rightarrow \quad f(x) = C \text{ constant.}$$

(Shows that there are no exotic functions that aren't constant, but that have zero derivative everywhere.)

COROLLARY.

If $f'(x) \equiv 0$ on (a, b) (i.e., $f'(x) = 0 \ \forall x \in (a,b)$)

Then $f(x)$ is constant on (a,b) (i.e., $\exists C \in \mathbb{R}$ s.t. $f(x) \equiv C$)

Proof. Let $x \in (a, b]$ be arbitrary, but fixed.

NOTE: If we prove something about x, then since x was arbitrary in $(a, b]$, that "something" holds for all x in $(a, b]$.

Since f is diff'ble $\Rightarrow$ cts. on (a,b), and since we assume f is right cts. at a, we can apply the MVT on the interval $[a, x]$, finding

$$f'(c) = \frac{f(x) - f(a)}{x - a}, \quad \text{for some } c \in (a, x).$$

But we know $f'(c) = 0$, as $c \in (a, x)$, so we must have $f(x) = f(a)$. This holds for all $x \in (a, b]$, so f is constant on $[a, b]$. $\square$

COROLLARY. Suppose f, g are cts. on $[a,b]$, and that for all $x \in (a,b)$, $\boxed{f'(x) = g'(x)}$.

Then f and g differ by a constant on $[a,b]$ — i.e., $\underbrace{\exists}_{\text{there exists}} K \in \mathbb{R}$ $\underbrace{\text{s.t.}}_{\text{such that}}$ $\underbrace{\forall}_{\text{for all}} x \in [a,b]$, $f(x) = g(x) + K$.

$f(x) = g(x) + K \Rightarrow f'(x) = g'(x)$
$\Leftarrow$

Proof. Let $h(x) := f(x) - g(x)$.

Then $h'(x) = f'(x) - g'(x) = 0$ for $x \in (a,b)$.

So, by Corollary 1, $h(x) = K$ constant on $[a,b]$ —

That is, $f(x) - g(x) = K$

$\Rightarrow$ $f(x) = g(x) + K$ for $x \in [a,b]$. □

Example. 5, p.240

If $f'(x) = 6e^{2x}$ and $f(0) = 7$, what is $f(x)$?

Observe: $\frac{d}{dx}\left[3e^{2x}\right] = 3\,\frac{d}{dx}\left[e^{2x}\right] = 3e^{2x}\frac{d}{dx}[2x] = 6e^{2x}$.

So we know that $\boxed{f(x) = 6e^{2x} + K}$, some $K \in \mathbb{R}$. Let's use $f(0) = 7$ to find K:

$$f(0) = 6e^{2(0)} + K = 7$$

$$\Rightarrow 6 + K = 7$$

$$\Rightarrow K = 1.$$

So $\boxed{f(x) = 6e^{2x} + 1.}$

Increasing and decreasing functions ...

$f(x)$ is INCREASING on the intervals where $f'(x) > 0$

$f(x)$ is DECREASING ——"—— $f'(x) < 0$.

$\boxed{\text{Example. } 6, \text{ p. } 240}$ Where is $f(x) = x^2 - 4x + 5$ increasing? Where is it decreasing?

Well, $f'(x) = 2x - 4$, and $f'(x) > 0$ when $x > 2$,

whereas $f'(x) < 0$ when $x < 2$.

So, f is increasing on $(-\infty, 2) = \{x \in \mathbb{R} : x < 2\}$

f is decreasing on $(2, +\infty) = \{x \in \mathbb{R} : x > 2\}$.

$\boxed{\text{Example. } 7, \text{ p. } 241}$ Prove $e^x + x - 2 = 0$ has <u>exactly</u> one real sol'n.

First, we show it has one sol'n, using the IVT. Let $f(x) = e^x + x - 2$. This is cts. everywhere, in particular on $[0,1]$, and $f(0) = -1 < 0$, but $f(1) = e - 1 > 0$, so $f(x) = 0$ has a sol'n $x_* \in (0,1)$; i.e., $e^{x_*} + x_* - 2 = 0$

Now, we show it cannot have <u>more</u> than one sol'n.
Note $f'(x) = e^x + 1$, and as $e^x > 0$ $\forall x$, we have $f'(x) > 0$ $\forall x$.
Thus, f is always increasing — so, if $x < x_*$, then $f(x) < f(x_*) \Rightarrow f(x) < 0$, and if $x > x_*$, then $f(x) > f(x_*) = 0$, so $f(x) \neq 0$ for $x > x_*$ or for $x < x_*$. Thus x_* is the only zero of f.

Example. 8, p. 241

Determine the open intervals where the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

is increasing, and those on which it is decreasing.

Well, $f'(x) = \frac{d}{dx}\left[3x^4 - 4x^3 - 12x^2 + 5\right]$

$$= 12x^3 - 12x^2 - 24x$$

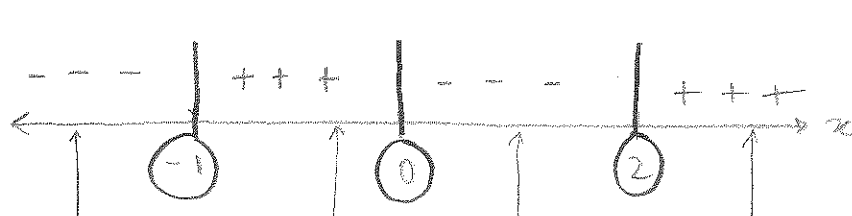
$$= 12x\,(x^2 - x - 2)$$

$$= \underbrace{12x}\,\underbrace{(x-2)}\,\underbrace{(x+1)}.$$

So $f'(x) = 0$ when: ① $12x = 0 \Leftrightarrow x = 0$

② $x - 2 = 0 \Leftrightarrow x = 2$

③ $x + 1 = 0 \Leftrightarrow x = -1$.



TEST $x = -2$
$f'(-2) = (-)(-)(-) < 0$

TEST $x = -0.5$
$f'(-0.5) = (-)(-)(+) > 0$

TEST $x = 1$
$f'(1) = (+)(-)(+) < 0$

TEST $x = 3$
$f'(3) = (+)(+)(+) > 0$

TEST THE SIGN OF f' — NOT OF f ITSELF

So: INC. on $(-1, 0) \cup (2, +\infty)$, DEC. on $(-\infty, -1) \cup (0, 2)$.

So, to determine where f is increasing/decreasing:

STEPS

1. FIND $f'(x)$
2. FIND THE POINTS WHERE $f'(x) = 0$
3. TEST THE SIGN OF $f'(x)$ AT POINTS BETWEEN THOSE VALUES
4. f IS INCREASING WHERE $f'(x) > 0$

   DECREASING WHERE $f'(x) < 0$.