CHAPTER 1: REVIEW

Understanding: Concepts and Definitions

Refer to the listed pages to review the concepts and definitions in this chapter that you need to understand.

Section		Pages
1.1	The definition of a function	2
	The domain and range of a function	2
	Dependent and independent variables	3
	Open and closed interval notation	4
	~~What is a formula vs. what is a relation~~	~~5~~
	The idea of a mathematical model	7
1.2	Slope-intercept and point-slope equations of straight lines	12
	The graph of an equation	13
	~~Circles and translates of graphs~~	~~13–14~~
	The graph of a function	15
	~~The vertical line test for graphs of functions~~	~~15~~
	Discontinuities of functions	16
	Parabolas and graphs of quadratic functions	17–18
	Graphic, numeric, and symbolic representations of functions	20
1.3	The definition of a power function	24
	Algebraic combinations of functions	26
	The definition of a polynomial	27
	The definition of a rational function	30
	The definition of an algebraic function	31
1.4	The sine and cosine functions and their graphs	35
	The definition of the composition of two functions	37
Covered in Ch. 3	~~The definition of an exponential function~~	~~38~~
	~~The definition of a logarithmic function~~	~~40~~

Objectives: Methods and Techniques

Work the listed problems in each section to practice the methods and techniques in this chapter that you need to master.

Section		Problems
1.1	Simplifying functional expressions	13, 15
	Finding the domain of a function defined by a formula	25, 29, 33
	Writing formulas for functions described verbally	37, 39, 41, 43, 45
	~~Numerical solution of equations by repeated tabulation~~	~~59, 60~~
1.2	Writing the equation of a given straight line	1, 5, 9
	~~Sketching the graph of a circle with given equation~~	~~13, 15~~
	Sketching a parabola with given equation	19
	Identifying and sketching the graph of a given function	33, 37, 39, 45, 49
	~~Algebraic and graphical investigation of high and low points~~	~~57, 61~~
1.3	Finding formulas for algebraic combinations of functions	1, 5
	Identifying the graph of a polynomial by determining its number of zeros and its behavior for $\|x\|$ large	7, 11
	Identifying the graph of a rational function by determining its asymptotes and its behavior for $\|x\|$ large	13, 15
	~~Finding graphically the number of real zeros of a polynomial~~	~~21, 23, 25, 39~~
1.4	Matching graphs and equations of trigonometric and exponential functions	1, 3, 5, 7
	Finding the formula for the composition $f(g(x))$ of two given functions f and g	11, 15, 17, 19
	~~Finding graphically the number of real solutions of a given transcendental equation~~	~~31, 33, 35, 39~~

CHAPTER 2: REVIEW

Understanding: Concepts and Definitions

Refer to the listed pages to review the concepts and definitions in this chapter that you need to understand.

Section		Pages
2.1	The relation between secant lines and tangent lines	55
	The difference quotient of a function f at a point $x = a$	57
	The slope of a tangent line as a limit of difference quotients	58
	The slope formula for the tangent line at a point on a parabola	58–59
	~~The relation between tangent and normal lines to a curve~~	~~59~~
2.2	The slope at $(a, f(a))$ as a limit as either $h \to 0$ or $x \to a$	64
	The idea of the limit of $f(x)$ as $x \to a$	65
	The constant, sum, product, quotient, and root laws of limits	68–69
	The substitution law and limits of compositions	71
	~~The four-step process for finding slope-predictor functions~~	~~72~~
2.3	The basic trigonometric limit	77
	The squeeze law of limits	78
	Right-hand and left-hand limits	79–80
	The relation between one-sided and two-sided limits	80
	Existence of tangent lines	81
	Infinite limits of functions	82
	The precise definition of the limit	85
2.4	Continuity of a function at a point	91
	Removable discontinuities of functions	92
	Continuity of combinations, polynomials and rational functions	93
	Continuity of trigonometric functions	93
	Continuity of compositions of continuous functions	94
	Continuity of a function on a closed interval	96
	The intermediate value property of continuous functions	97
	Existence of solutions of equations	97

LIMITS

CONT.

Objectives: Methods and Techniques

Work the listed problems in each section to practice the methods and techniques in this chapter that you need to master.

Section		Problems
2.1	~~Finding the equation of the tangent line at a point on a parabola~~	~~9, 11~~
	~~Find the point(s) on a curve where the tangent line is horizontal~~	~~17, 21~~
	Finding equations of ~~both~~ tangent ~~and normal~~ lines to a curve	25, 27
	~~Solving applied problems by finding high points on parabolas~~	~~29, 31~~
	~~Numerically investigating the slope of a tangent line at a point~~	~~37, 41, 45~~
2.2	Using limit laws to evaluate limits of functions	3, 7, 11
	Finding limits of quotients after algebraic simplification	21, 25, 31, 35
	~~Using the four-step process to find a slope-predictor function~~	~~37, 41, 45~~
	~~Investigating a limit numerically~~	~~47, 49, 55~~
2.3	Using limit laws to evaluate trigonometric limits	1, 3, 9, 11, 13, 25
	Using the one-sided limit laws to evaluate limits	29, 35, 39, 43, 45
	~~Determining behavior where one-sided limits fail to exist~~	~~49, 51, 55~~
	Using the precise definition to establish a limit	75, 79
2.4	Using limit laws to establish continuity of functions	3, 5, 7, 9, 11, 13
	Determining where a given function is continuous	17, 21, 23, 25, 31
	Determining whether or not a discontinuity is removable	37, 39, 43, 45, 47
	Applying the intermediate value property to locate solutions	53, 55
	~~Numerical investigation of continuity at a given point~~	~~73, 75~~

$\varepsilon - \delta$

CHAPTER 3: REVIEW

Understanding: Concepts, Definitions, Formulas

Refer to the listed pages to review the concepts, definitions, and formulas in this chapter that you need to understand.

$\cancel{D_x[f(x)]}$

$\frac{d}{dx}[f(x)]$

$f'(x)$

$\frac{dy}{dx}$

CHAPTER 3: REVIEW (Continued)

Objectives: Methods and Techniques

Work the listed problems in each section to practice the methods and techniques in this chapter that you need to master.

Section		Problems
3.1	~~Using a differentiation rule to differentiate quadratic functions~~	~~5, 9~~
	Applying the definition of the derivative to find $f'(x)$	13, 17, 19
	~~Finding when the velocity of a moving particle is zero~~	~~25, 29, 29~~
	Matching the graphs of a function and its derivative	31, 33
	~~Calculating the rate of growth of a population~~	~~41, 53~~
	~~Calculating rates of change in geometric situations~~	~~45, 47, 49~~
3.2	Applying general differentiation rules to find derivatives	3, 5, 9, 11, 15, 19, 21, 27, 35
	Finding tangent lines to graphs	43, 45, 49
	~~Calculating rates of change in geometric situations~~	~~51, 53~~
3.3	Using the chain rule to differentiate functions	3, 5, 9, 13, 15, 23, 25, 29, 33, 35
	~~Calculating rates of change in geometric situations~~	~~49, 51, 53, 57, 59~~
3.4	Using rules to differentiate algebraic functions	3, 5, 9, 13, 17, 21, 23, 29, 35, 41
	~~Finding tangent lines to graphs of algebraic functions~~	~~47, 49, 53~~
	Matching the graphs of a function and its derivative	57, 59
3.5	Finding the maximum and minimum values of a function defined on a closed interval	~~5, 7, 11,~~ 15, 19, 25, 33, 35, 37
	Matching the graphs of a function and its derivative	49, 51
3.6	Solving applied maximum-minimum problems	~~3, 5, 7,~~ 11, 17, 21, 23, 27, 31, 33, 45
3.7	Calculating derivatives of trigonometric functions	5, 7, 9, 13, 15, 21, 27, 35, 39, 45, 47, 51, 53
	~~Finding tangent lines to trigonometric graphs~~	~~61, 65~~
	~~Solving trigonometric rate of change problems~~	75, 77
	~~Solving trigonometric maximum-minimum problems~~	~~79, 81, 83~~
3.8	Calculating derivatives of exponential and logarithmic functions	5, 7, 11, 15, 19, 23, 27, 29, 31, 33
	~~Applying laws of logarithms before differentiating~~	~~39, 41~~
	Finding a derivative by logarithmic differentiation	49, 51
	~~Finding tangent lines to exponential graphs~~	59, 61
~~3.9~~	~~Finding derivatives and tangent lines by implicit differentiation~~	~~3, 7, 15, 21, 25~~
	~~Solving applied related-rates problems~~	~~37, 39, 41, 43, 45, 47, 51, 53, 55, 61~~
~~3.10~~	~~Applying Newton's method to find a solution of an equation~~	~~3, 5, 9, 15, 17, 27, 33~~

MISCELLANEOUS PROBLEMS

Find dy/dx in Problems 1 through 35.

1. $y = x^2 + \dfrac{3}{x^2}$

2. $y^2 = x^2$

3. $y = \sqrt{x} + \dfrac{1}{\sqrt[3]{x}}$

4. $y = (x^2 + 4x)^{5/2}$

5. $y = (x-1)^7(3x+2)^9$

6. $y = \dfrac{x^4 + x^2}{x^2 + x + 1}$

7. $y = \left(3x - \dfrac{1}{2x^2}\right)^4$

8. $y = x^{10}\sin 10x$

9. $xy = 9$

10. $y = \sqrt{\dfrac{1}{5x^6}}$

11. $y = \dfrac{1}{\sqrt{(x^3 - x)^3}}$

12. $y = \sqrt[3]{2x+1}\,\sqrt[5]{3x-2}$

13. $y = \dfrac{1}{1+u^2}$ where $u = \dfrac{1}{1+x^2}$

14. $x^3 = \sin^2 y$

15. $y = \left(\sqrt{x} + \sqrt[3]{2x}\right)^{7/3}$

16. $y = \sqrt{3x^5 - 4x^2}$

17. $y = \dfrac{u+1}{u-1}$, where $u = \sqrt{x+1}$

18. $y = \sin(2\cos 3x)$

19. $x^2y^2 = x + y$

20. $y = \sqrt{1 + \sin\sqrt{x}}$

21. $y = \sqrt{x + \sqrt{2x + \sqrt{3x}}}$

TODAY: Midterm Exam Review sheet
Midterm review problems.

EXAM DETAILS.

- No books, notes, calculators
- 2 hours
- 9 problems + 1 bonus

This is useful information! We can create our own review sheet.

So, what's left:

Ch. 1: Most of this material will be covered 'en passant' – in passing – so, as parts of "bigger" problems, but what's most important / useful in this context:

- Finding domains
- Sketching graphs – simple problems only.
- Function behavior for $|x|$ large, i.e., $\lim_{x\to+\infty} f(x)$, $\lim_{x\to-\infty} f(x)$.

Ch. 2: • Limits

- Epsilon/delta def'n – use this to prove a limit's value
- Limit laws (sum, product, quotient, composition)
- Squeeze theorem
- Def'n of the derivative

• Continuity

- Def'n
- Classifying discontinuities
- IVT

Ch. 3 : The big chapter! - DERIVATIVES

- Def'n of derivative : $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

- Characterization / explanation of the derivative

  - Slope of the tangent line (write the eq'n of a tan. line at a point)
  - Instantaneous rate of change
  - Increasing / decreasing fns. have positive/negative deriv.

- Notations — $f'(x)$ $\quad \dfrac{dy}{dx}$ $\quad \dfrac{d}{dx}\left[\quad\right]$

  $f'(c)$ $\quad \left.\dfrac{dy}{dx}\right|_{x=c}$

- Know the def'ns of velocity, acceleration and/or jerk factor

- LAWS

  - Linear combinat'n : $\frac{d}{dx}\left[a f(x) + b g(x)\right] = a f'(x) + b g'(x)$
  - Power rule : $\frac{d}{dx}\left[x^r\right] = r x^{r-1}$, $0 \neq r \in \mathbb{Q}$.
  - Product rule : $\frac{d}{dx}\left[f(x) g(x)\right] = f'(x) g(x) + g'(x) f(x)$
  - Quotient rule : $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \dfrac{g(x) f'(x) - f(x) g'(x)}{[g(x)]^2}$
  - Chain rule : $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x)$
  - Trig fns : $\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$, $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$
  - Exponentials : $\frac{d}{dx}\left[a^x\right] = \ln(a)\, a^x$; $\frac{d}{dx}\left[e^x\right] = e^x$
  - Logarithms : $\frac{d}{dx}\left[\log_b(x)\right] = \dfrac{1}{\ln(b)\, x}$; $\frac{d}{dx}\left[\ln(x)\right] = \dfrac{1}{x}$.

Ch. 3: Derivatives, ct'd.

- Theorem: diff'ble $\Rightarrow$ cts.

- Vertical tangent lines & undefined derivatives

- Theorem: derivatives of inverse fns.

- Logarithmic differentiation

- Optimization
  - Over a closed interval
  - Global optimization.

- Matching a graph to its derivative.

Let's do some example problems for the rest of class.

Lecture 6: Example problems

1.MP.6 Domain of defn: $f(x) = \dfrac{x+1}{x^2-2x} = \dfrac{x+1}{x(x-2)}$

Defined when $x \neq 0$ and $x - 2 \neq 0 \Leftrightarrow x \neq 2$.

So $D = \{x \in \mathbb{R} : x \neq 0 \text{ and } x \neq 2\}$

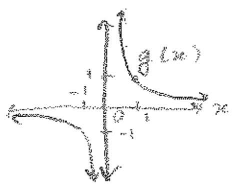
$= (-\infty, 0) \cup (0, 2) \cup (2, +\infty)$

Either set notation or interval notation is appropriate — MUST USE CORRECTLY FOR EXAM.

$(-\infty, 0) \cup \{1\} \cup (2, +\infty)$

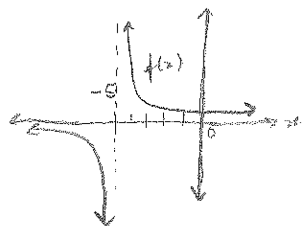
[1]

1.MP.41 Graph $f(x) = \dfrac{1}{x+5}$. Know what $g(x) = \dfrac{1}{x}$ looks like:

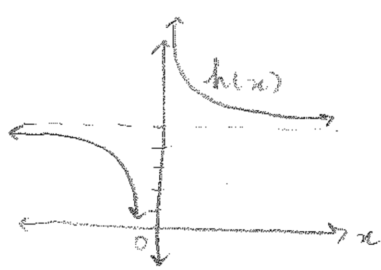


Adding 5 to the <u>argument</u> shifts the graph to the left by 5. (Try a test point — $f(1) = \frac{1}{6} = g(6)$, so we obtain f by shifting g to the <u>left</u>.)

So f looks like:



Similar: $h(x) = \dfrac{1}{x} + 5$. Shift <u>up</u> by 5 units:



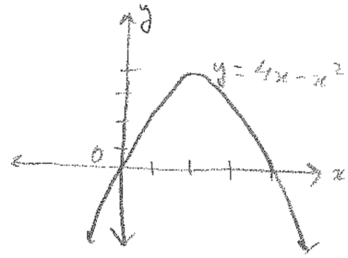
1.MP. 40. Graph $y = 4x - x^2$.

Completing square: $4x - x^2 = -(x-2)^2 + 4 \Rightarrow y - 4 = -(x-2)^2$.

So, vertex is at $(2, 4)$, parabola points downward and is not stretched / compressed.

Note: $y = x(4-x)$ implies zeros are at $x = 0$, $x = 4$.

The graph:



Note the limiting behavior: $f(x) \xrightarrow[|x| \to +\infty]{} -\infty$.

2.3.76. Prove (ε-δ) that $\lim_{x \to 5} \underbrace{(17x - 35)}_{f(x)} = 50$.

Let $\varepsilon > 0$ be fixed. Want to find $\delta > 0$ s.t. $\boxed{|x - 5| < \delta \Rightarrow |f(x) - 50| < \varepsilon}$.

Well, look: $|f(x) - 50| = |17x - 35 - 50| = |17x - 85|$

$= 17|x - 5|$.

So let $\delta < \frac{\varepsilon}{17}$. Then if $|x - 5| < \delta = \frac{\varepsilon}{17}$, we have

$|f(x) - 50| = 17|x - 5| < 17\left(\frac{\varepsilon}{17}\right) = \varepsilon$. Great. Final writeup:

> Let $\varepsilon > 0$ be fixed, and let $\delta < \frac{\varepsilon}{17}$. Suppose that $|x - 5| < \delta$, and see: $|f(x) - 50| = |17x - 35 - 50| = |17x - 85| = 17|x - 5| < 17\left(\frac{\varepsilon}{17}\right) = \varepsilon$.
>
> Thus, $\lim_{x \to 5} (17x - 35) = 50$.

2.2.4. $\lim_{x\to -2} (x^3-3x+3)(x^2+2x+5) = \left[\lim_{x\to -2} x^3-3x+3\right]\left[\lim_{x\to -2} x^2+2x+5\right]$ 13

by the product rule, if both limits on the RHS exist.

NOW, $\lim_{x\to -2} x^3-3x+3 \overset{\text{SUM}}{=} \left[\lim_{x\to -2} x^3\right] - 3\left[\lim_{x\to -2} x\right] + 3\left[\lim_{x\to -2} 1\right]$

ROOT LAW CONST. LAW

$= \left[\lim_{x\to -2} x\right]^3 - 3(-2) + 3$

PROD. LAW – if $\lim_{x\to -2} x$ exists, which it does

$= (-2)^3 - 3(-2) + 3 = -8+6+3 = 1.$

and $\lim_{x\to -2} x^2+2x+5 = \left[\lim_{x\to -2} x^2\right] + 2\left[\lim_{x\to -2} x\right] + 5\left[\lim_{x\to -2} 1\right]$

$= \left[\lim_{x\to -2} x\right]^2 + 2(-2) + 5$

$= (-2)^2 + 2(-2) + 5 = 4-4+5 = 5$

So, the first product rule holds, and

$\lim_{x\to -2} (x^3-3x+3)(x^2+2x+5) = 1(5) = 5.$

If we hadn't been spec<u>i</u>fically asked for limit laws...

We know that $(x^3-3x+3)(x^2+2x+5)$ is a polynomial, and we know that polynomials are diff'ble everywhere (we even have a formula for the derivative). Well, differentiability implies continuity, and for cts. functions, the limit can be evaluated by substitution. So $\lim_{x\to -2} (x^3-3x+3)(x^2+2x+5) =$
$= ((-2)^3-3(-2)+3)((-2)^2+2(-2)+5) = 1(5) = 5.$

(YES, you need to write the whole paragraph!!)

2.3.28 Find $\lim_{x\to 0} \sqrt[3]{x}\ \sin\left(\frac{1}{x}\right)$. 4

Know that $-\sqrt[3]{x} \le \sqrt[3]{x}\ \sin\left(\frac{1}{x}\right) \le \sqrt[3]{x}$, since $-1 \le \sin\left(\frac{1}{x}\right) \le 1$

for all $x \neq 0$. Furthermore,

$$\lim_{x\to 0} -\sqrt[3]{x} \underset{\text{CONST. MULT. LAW}}{=} -\lim_{x\to 0} \sqrt[3]{x} \underset{\text{ROOT LAW}}{=} -\sqrt[3]{0} = 0$$

and $\lim_{x\to 0} \sqrt[3]{x} = \sqrt[3]{0} = 0$ (Again, root law.)

Thus, by the Squeeze Theorem, $\lim_{x\to 0} \sqrt[3]{x}\ \sin\left(\frac{1}{x}\right) = 0$.

NOTES: • Find g, h s.t. $g(x) \le f(x) \le h(x)$

AND

$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) =: L$

• Then say "by the squeeze theorem, $\lim_{x\to a} f(x) = L$".

Limits & def'n of derivative. 5

Find $\lim_{x \to 0} \frac{(4-x)^2 - 4^2}{x}$.

Realize: If $g(t) = t^2$, then this limit is just the defin of the derivative, evaluated at $t = 4$.

We know $g(t) = t^2$ is diff'ble at $t=4$, so the limit exists, and we calculate its value by using our derivative laws:

$$\lim_{x \to 0} \frac{(4-x)^2 - 4^2}{x} = \frac{d}{dt}\left[t^2\right]\Big|_{t=4} = \left[2t\right]\Big|_{t=4} = 8.$$

2.4.13 Show $f(x) = \frac{x}{\cos x}$ is cts. on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

① Note that $f(x)$ is not defined when $\cos x = 0$, i.e., at $\pm \frac{(2n+1)\pi}{2}$ for $n \in \mathbb{N}$. The interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ includes none of these points, so f is defined everywhere there.

② Note that $\lim_{x \to a} \frac{x}{\cos x} = \frac{\lim_{x \to a} x}{\lim_{x \to a} \cos x}$, if $\lim_{x \to a} \cos x \neq 0$. This is true, though, since $a \neq \pm \pi/2$. So, $\lim_{x \to a} \frac{x}{\cos x} = \frac{\lim_{x \to a} x}{\lim_{x \to a} \cos x} = \frac{a}{\cos a}$.

③ Moreover, $\lim_{x \to a} \frac{x}{\cos x} = \frac{a}{\cos a} = f(a)$, for all $a \in (-\pi/2, \pi/2)$

$f(x)$ is thus cts. on $(-\pi/2, \pi/2)$.