Lecture 5: Example problems.

3.8.52 $y = (3+2^x)^x$. Find $\frac{dy}{dx}$. We use logarithmic differentiation:

① $\ln y = \ln\left[(3+2^x)^x\right]$
$= x \ln(3+2^x)$.

② $\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[x \ln(3+2^x)\right]$

$= x \frac{d}{dx}\left[\ln(3+2^x)\right] + \ln(3+2^x)\frac{d}{dx}[x]$

$= x \frac{\frac{d}{dx}\left[3+2^x\right]}{3+2^x} + \ln(3+2^x)$

$= \frac{x \ln(2)\, 2^x}{3+2^x} + \ln(3+2^x)$

③ $\frac{dy}{dx} = y\left[\frac{x\ln(2)2^x}{3+2^x} + \ln(3+2^x)\right]$

$= (3+2^x)^{x-1}\left[\frac{x\ln(2)2^x + (3+2^x)\ln(3+2^x)}{3+2^x}\right]$

$= (3+2^x)^{x-1}\left[x\ln(2)2^x + (3+2^x)\ln(3+2^x)\right]$.

3.4.57-62 (see book). Match up critical points on graphs,
— " — intervals where f is inc/dec.

An example abt. rockets (something like WebWork prob. 5):

A model of the position of a rocket is

$x(t) = 0.01t^4 - 0.8t^3 + 15t^2 - 8t + 20.$ (feet) (t: sec)

Find the min/max values of the acceleration for $t \in [0,100]$.

Recall: velocity $:= v(t) = x'(t) = 0.04t^3 - 2.4t^2 + 30t - 8$
acceleration $:= a(t) = v'(t) = x''(t) = 0.12t^2 - 4.8t + 30$
"jerk factor" $:= a'(t) = x'''(t) = 0.24t - 4.8$.

Critical points of $a(t)$ are when $a'(t) = 0$ or is undefined.
Can see $a'(t)$ never undefined, and is zero at $t = 4.8/0.24 = 20$ sec.
Test: $a(0) = 30$ $a(20) = -18$ $a(100) = 750$.
min. (at $a(20)$) max. (at $a(100)$)

example probs, ct'd.

Ex: Find the global extrema of $g(t) = 4te^{-t}$, $t > 0$

$g'(t) = \frac{d}{dt}[4t]e^{-t} + 4t\frac{d}{dt}[e^{-t}]$

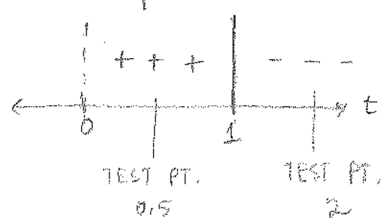
$= 4e^{-t} + 4t(-e^{-t}) = 4e^{-t} - 4te^{-t} = 4e^{-t}(1-t)$. $= \frac{4(1-t)}{e^t}$

Critical points occur when $g'(t) = 0$ or when $g'(t)$ undefined.
See: $g'(t)$ is never undefined.

Now, $g'(t) = 0$ when $4e^{-t}(1-t) = 0$ – i.e., when

either $\underbrace{\text{(I)}\ 4e^{-t} = 0}_{\text{never happens (check graph if this isn't obvious)}}$ or $\underbrace{\text{(II)}\ 1-t = 0}_{\text{occurs when } t=1.}$

The only critical value is $t = 1$, so inspect the derivative using test points:



So, derivative goes from positive to negative at $t = 1$, meaning the fn. increases until $t = 1$ and then decreases – so have a local max. at $t = 1$, i.e., at $(1, g(1)) = (1, 4/e)$.

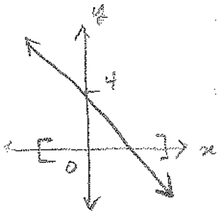
Is it a global max.? – Check the behavior of $g(t)$ as $t \to 0^+$ and as $t \to +\infty$.

$\lim_{t \to 0^+} 4te^{-t} = 0$, and $\lim_{t \to +\infty} 4te^{-t} = 0$ (check the graph for this limit)

So, the local max. is a global max., and since $t = 0$ was not in our domain, f never achieves its global min.

3.5.12. $f(x) = 4 - 3x$ over $[-1, 5]$. LOCAL OPTIMIZATION

$f'(x) = -3$ is never zero nor undefined, so there are no critical points. Check only endpoints:



$f(-1) = 7$, $f(5) = -11$. Thus, $(-1, 7)$ is the max., $(5, -11)$ the min.

3.5.20 $f(x) = x^2 + \frac{16}{x}$ over $[1, 3]$. LOCAL OPTIMIZATION

Then $f'(x) = 2x - \frac{16}{x^2} = \frac{2x^3 - 16}{x^2}$.

$f'(x)$ is undefined when $x = 0$
$f'(x) = 0$ when $2x^3 - 16 = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$.

So, critical points are 0 and 2, and 0 is not in our domain of interest. So check only 2 and endpoints:

$f(1) = 17$
$f(2) = 12$
$f(3) = 9 + \frac{16}{3} = \frac{27 + 16}{3} = \frac{43}{3} = 14\tfrac{1}{3}$.

Thus, $(2, 12)$ is the min. and $(1, 17)$ is the max.

Example Probs, lec. 5.

3.6.2

Perimeter $= 2(b+h) = 200$ m

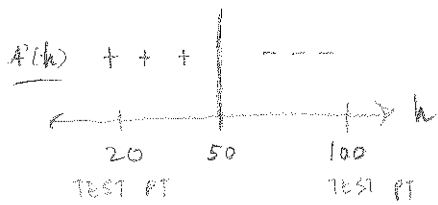
Area $= bh$. Find max area.

Well, $2(b+h) = 200 \Rightarrow b = 100 - h$, so plug in to area:

$A(h) = (100-h)h = 100h - h^2$ Maximize this.

$A'(h) = 100 - 2h$ is always defined, and is 0 when $h = 50$.
So A has a critical point at $h = 50$.

GLOBAL OPTIMIZATION



So, the fn. has a local max. at $x = 50$. Is it a global max? Check limiting behavior:

$\lim_{h \to -\infty} A(h) = -\infty$, $\lim_{h \to +\infty} A(h) = -\infty$

So, indeed, $h = 50$ at the global max $(50, A(50)) = (50, 50^2)$

LOCAL OPTIMIZAT'N

Could do another way, if we'd realized that $h > 0$ and $A > 0$ are physical limitations that allow us to restrict the domain to $[0, 100]$.

Critical point: $h = 50$; endpoints: $h = 0$, $h = 100$.

$A(0) = \underline{0}$ $A(50) = \underline{50^2}$ $A(100) = \underline{0}$

So max. over $[0, 100]$ is $(50, 50^2)$.