Lecture 7: Example problems.

3.9.4 $x^3+y^3=1. \Rightarrow \frac{d}{dx}\left[x^3+y^3\right]=\frac{d}{dx}[1]$

$$LHS = \frac{d}{dx}[x^3]+\frac{d}{dx}[y^3] \qquad RHS = 0$$

$$= 3x^2 + 3y^2\frac{dy}{dx}$$

$$\Rightarrow 3x^2+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow \boxed{\frac{dy}{dx}=-\frac{x^2}{y^2}.}$$

Or, $y=(1-x^3)^{1/3} \Rightarrow \frac{dy}{dx}=\frac{1}{3}(1-x^3)^{-2/3}(-3x^2)$

$$= \frac{-x^2}{(1-x^3)^{2/3}} = -\frac{x^2}{y^2} \quad ✓.$$

3.9.10 $x^5+y^5=5x^2y^2 \Rightarrow \frac{d}{dx}\left[x^5+y^5\right]=\frac{d}{dx}\left[5x^2y^2\right].$

$$LHS = 5x^4+5y^4\frac{dy}{dx}$$

$$RHS = 10xy^2+10x^2y\frac{dy}{dx}$$

so, $5x^4+5y^4\frac{dy}{dx}=10xy^2+10x^2y\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx}\left[y^4-2x^2y\right]=2xy^2-x^4$$

$$\Rightarrow \boxed{\frac{dy}{dx}=\frac{2xy^2-x^4}{y^4-2x^2y}.}$$

3.9.12 $\cos(x+y) = \sin x \sin y \Rightarrow \frac{d}{dx}[\cos(x+y)] = \frac{d}{dx}[\sin x \sin y]$

$$LHS = -\sin(x+y)\frac{d}{dx}[x+y]$$
$$= -\sin(x+y)\left(1+\frac{dy}{dx}\right)$$

$$RHS = \cos x \sin y + \cos y \sin x \frac{dy}{dx}$$

$$\Rightarrow -\sin(x+y)\left(1+\frac{dy}{dx}\right) = \cos x \sin y + \cos y \sin x \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[\cos y \sin x + \sin(x+y)\right] = -\sin(x+y) - \cos x \sin y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin(x+y) - \cos x \sin y}{\cos y \sin x + \sin(x+y)}$$

$$= \boxed{\frac{-\sin x \cos y - 2\cos x \sin y}{2\cos y \sin x + \sin y \cos x}}$$

3.9.14 $xy = e^{-xy} \Rightarrow \frac{d}{dx}[xy] = \frac{d}{dx}[e^{-xy}]$

$$LHS = x\frac{dy}{dx} + y$$

$$RHS = e^{-xy}\frac{d}{dx}[-xy]$$
$$= e^{-xy}\left[-\left(x\frac{dy}{dx}+y\right)\right]$$

So, $x\frac{dy}{dx} + y = -e^{-xy}\left[x\frac{dy}{dx} + y\right]$

$$\Rightarrow x\frac{dy}{dx}\left[1+e^{-xy}\right] = -y\left(1+e^{-xy}\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-y}{x}}$$

$\boxed{3.9.24}$ $xy = 6e^{2x-3y}$; find eq'n of line tan at $(3,2)$

$$\Downarrow$$

$$\frac{d}{dx}[xy] = \frac{d}{dx}\left[6e^{2x-3y}\right]$$

$$\left.\begin{aligned} LHS &= x\frac{dy}{dx} + y \\ RHS &= 6e^{2x-3y}\frac{d}{dx}[2x-3y] \\ &= 6e^{2x-3y}\left[2 - 3\frac{dy}{dx}\right] \end{aligned}\right\} \Rightarrow x\frac{dy}{dx} + y = 6e^{2x-3y}\left(2 - 3\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(x + 18e^{2x-3y}\right) = 12e^{2x-3y} - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{12e^{2x-3y} - y}{x + 18e^{2x-3y}}.$$

So, at $(3,2)$, $\left.\frac{dy}{dx}\right|_{(3,2)} = \frac{12e^{2(3)-3(2)} - 2}{3 + 18e^{2(3)-3(2)}} = \frac{12-2}{3+18} = \frac{10}{21}$.

Thus, the eq'n of the tan. line — which passes through the pt. $(3,2)$ and has slope $\frac{10}{21}$ — is:

$$y - 2 = \frac{10}{21}(x-3) \Rightarrow \boxed{y = \frac{10}{21}x + \frac{4}{7}.}$$

$\boxed{3.9.26}$ $x^2 - xy + y^2 = 19$ at $(3,-2)$.

$$\Downarrow$$

$$\frac{d}{dx}[x^2 - xy + y^2] = \frac{d}{dx}[19]$$

$$\left.\begin{aligned} LHS &= 2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} \\ RHS &= 0 \end{aligned}\right\} \Rightarrow 2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

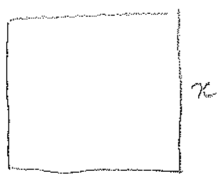
$$\Rightarrow \frac{dy}{dx}[2y - x] = y - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-2x}{2y-x}$$

So, $\left.\frac{dy}{dx}\right|_{(3,-2)} = \frac{-2-2(3)}{2(-2)-3} = \frac{-8}{-7} = \frac{8}{7}$.

So the line is $y + 2 = \frac{8}{7}(x-3) \Rightarrow \boxed{y = \frac{8}{7}x - \frac{38}{7}}$

3.9.54 A square is expanding. When each edge is 10 cm, its area is inc. at 120 cm²/sec. At what rate is the length of each edge changing then?



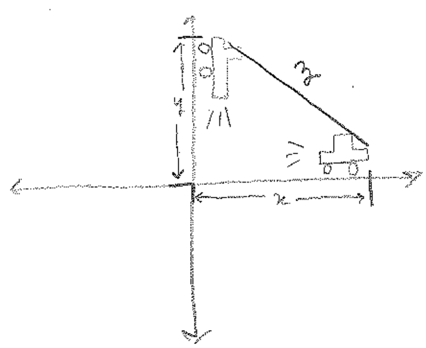
$A(x) = x^2$. Know $\left.\dfrac{dA}{dt}\right|_{t=T} = 120 \dfrac{cm^2}{sec}$, $x(T) = 10$ cm

Want to find $\left.\dfrac{dx}{dt}\right|_{t=T}$.

Differentiate: $\dfrac{d}{dt}[A] = \dfrac{d}{dt}[x^2] \Rightarrow \dfrac{dA}{dt} = 2x\dfrac{dx}{dt} \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{2x}\dfrac{dA}{dt}$.

Substitute: $\left.\dfrac{dx}{dt}\right|_{t=T} = \dfrac{1}{2x(T)}\left.\dfrac{dA}{dt}\right|_{t=T} = \dfrac{1}{2(10 \text{ cm})}\left(120 \dfrac{cm^2}{sec}\right) = \boxed{6 \dfrac{cm}{sec}}$

3.9.56



Know (Pythagorean thm): $z^2 = x^2 + y^2$.

Know: $x(10) = 0$

$y(11) = 0$

$\dfrac{dx}{dt} = 30$ mi/hr const

$\dfrac{dy}{dt} = 40$ mi/hr const.

Thus, $x(t) = 30(t-10)$ and so $x(13) = 30(13-10) = 30(3) = 90$ mi

$y(t) = 40(t-11)$

$y(13) = 40(13-11) = 40(2) = 80$ mi.

$z(13) = \sqrt{90^2 + 80^2} \approx 120.4$

Want to find $\left.\dfrac{dz}{dt}\right|_{t=13}$.

Differentiate: $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} \Rightarrow \dfrac{dz}{dt} = \dfrac{1}{z}\left[x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right]$.

Substitute: $\left.\dfrac{dz}{dt}\right|_{t=13} = \dfrac{1}{z(13)}\left[x(13)\left.\dfrac{dx}{dt}\right|_{t=13} + y(13)\left.\dfrac{dy}{dt}\right|_{t=13}\right] = \dfrac{1}{120.4}\left[90(30) + 80(40)\right]$

$\approx \boxed{49 \text{ mi/hr}}$

$\boxed{3.10.6}$ $f(x) = x^3 + 4x - 1$, on $[0,1]$.

$$f'(x) = 3x^2 + 4 \Rightarrow x_{n+1} = x_n - \frac{x_n^3 + 4x_n - 1}{3x_n^2 + 4}$$

$$= \frac{3x_n^3 + 4x_n - x_n^3 - 4x_n + 1}{3x_n^2 + 4}$$

$$= \frac{2x_n^3 + 1}{3x_n^2 + 4}.$$

Start with $x_0 = \frac{1}{2}$.

Then $x_1 = \frac{2\left(\frac{1}{2}\right)^3 + 1}{3\left(\frac{1}{2}\right)^2 + 4} = \frac{5}{19} \approx 0.26315$

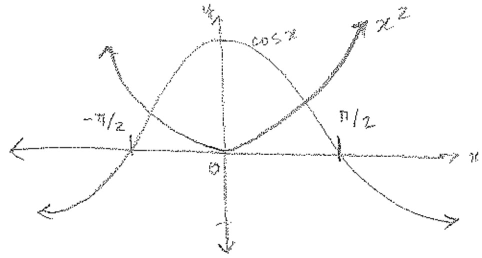
$$x_2 = \frac{2\left(\frac{5}{19}\right)^3 + 1}{3\left(\frac{5}{19}\right)^2 + 4} = \frac{7109}{28861} \approx 0.24631$$

$$x_3 = \frac{2\left(\frac{7109}{28861}\right)^3 + 1}{3\left(\frac{7109}{28861}\right)^2 + 4} = \cdots \approx \boxed{0.24626}$$

$$x_4 \approx \frac{2(0.24626)^3 + 1}{3(0.24626)^2 + 4} = \cdots \approx \boxed{0.24626}$$

Lec 7, probs cont'd

3.10.28 $\quad x^2 = \cos x$.



See: one root on $\left[-\frac{\pi}{2}, 0\right]$

and another on $\left[0, \frac{\pi}{2}\right]$.

So our initial guesses will be $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$ (do the iterations twice.)

Let $f(x) = x^2 - \cos(x)$, so $f'(x) = 2x + \sin(x)$.

Thus,
$$x_{n+1} = x_n - \frac{x_n^2 - \cos(x_n)}{2x_n + \sin(x_n)} = \frac{x_n^2 + x_n \sin(x_n) + \cos(x_n)}{2x_n + \sin(x_n)}$$

FIRST ROOT:
$$x_1 = \frac{\left(-\frac{\pi}{4}\right)^2 + \left(-\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{4}\right) + \cos\left(-\frac{\pi}{4}\right)}{2\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right)} \approx -0.825021$$

$$x_2 = \frac{x_1^2 + x_1 \sin(x_1) + \cos(x_1)}{2(x_1) + \sin(x_1)} \approx \boxed{-0.824132}2$$

$$x_3 \approx \boxed{-0.824132}23$$

SECOND ROOT:
$$x_1 = \frac{\left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)}{2\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)} \approx 0.825021$$

$$x_2 = \frac{x_1^2 + x_1 \sin(x_1) + \cos(x_1)}{2(x_1) + \sin(x_1)} \approx \boxed{0.824132}7$$

$$x_3 = \ldots \approx \boxed{0.824132}3$$