Lecture 8: example problems.

$\boxed{4.2.18}$ Find the linear approx. $L(x)$ to $f(x) = \frac{1}{\sqrt{1+x}}$ near $a = 0$.

Well, $f'(x) = \frac{d}{dx}(1+x)^{-1/2} = -\frac{1}{2}(1+x)^{-3/2}\frac{d}{dx}(1+x) = \frac{-1}{2(1+x)^{3/2}}$

so $f'(0) = \frac{-1}{(1+0)^{3/2}} = -1/2$

and $f(0) = \frac{1}{\sqrt{1+0}} = 1$

So $L(x)$ is a line w/slope -1 that passes through $(0,1)$.

$$L(x) - 1 = -\frac{1}{2}(x-0) \Rightarrow \boxed{L(x) = 1 - \frac{x}{2}}.$$

$\boxed{4.2.22}$ $f(x) = e^{-x} \Rightarrow f'(x) = e^{-x}\frac{d}{dx}(-x) = -e^{-x}$

so $f'(0) = -1$
and $f(0) = e^{-0} = 1$

So again, $L(x)$ has slope -1, passes through $(0,1)$

$\boxed{L(x) = 1 - x}$ as above.

$\boxed{4.2.24}$ $f(x) = \ln(1+x) \Rightarrow f'(x) = \frac{\frac{d}{dx}(1+x)}{1+x} = \frac{1}{1+x}$

so $f'(0) = \frac{1}{1+0} = 1$
and $f(0) = \ln(1+0) = 0$

So $L(x)$ has slope 1, passes through $(0,0)$

Thus $L(x) - 0 = 1(x-0) \Rightarrow \boxed{L(x) = x}.$

4.2.26 Approximate $\sqrt{102}$ using lin. approx.

Let $f(x) = \sqrt{x}$ and $a = 100$. Note $f(100) = 10$,

and $f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$.

Thus, $L(x)$ has slope $\frac{1}{20}$ and passes through $(100, 10)$.

$L(x) - 10 = \frac{1}{20}(x - 100) \Rightarrow L(x) = 5 + \frac{1}{20}x$

Thus, $L(102) = 5 + \frac{102}{20} = \frac{202}{20} = \frac{101}{10} = 10.1$,

So $\boxed{\sqrt{102} \approx 10.1}$ using local lin. approx.

4.2.32 $\sin 32^\circ$. Let $f(x) = \sin(x)$, and $a = 30^\circ$.

note that $\sin(30^\circ) = 0.5$ exactly; note also that

$f'(x) = \cos(x) \Rightarrow f'(30^\circ) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$ exactly.

Thus, $L(x)$ has slope $\frac{\sqrt{3}}{2}$, passes through $(30, 1/2)$.

$L(x) - \frac{1}{2} = \frac{\sqrt{3}}{2}(x - 30) \Rightarrow L(x) = \frac{1 - 30\sqrt{3} + \sqrt{3}x}{2}$

So $L(32) = \frac{1 - 30\sqrt{3} + 32\sqrt{3}}{2} = \frac{1 + 2\sqrt{3}}{2} = \frac{1}{2} + \sqrt{3}$

Thus, $\boxed{\sin(32^\circ) \approx \frac{1}{2} + \sqrt{3}}$ using local linear approx.

$\boxed{4.2.45}$ The range $R = \frac{1}{32} v^2 \sin 2\theta$, $\theta = 45°$ and v inc. from $80 \frac{ft}{sec}$ to $81 \frac{ft}{sec}$.

Well, find $L(v)$ for $R = \frac{1}{32} v^2 \sin(2\cdot 45) = \frac{1}{32} v^2 (1) = \frac{v^2}{32}$,
around $a = 80$ ft/sec $(v=a)$. Note, $R\Big|_{v=80} = \frac{80^2}{32} = 200$

$$\frac{dR}{dv} = \frac{2v}{32} = \frac{v}{16} \Rightarrow \frac{dR}{dv}\Big|_{v=80} = \frac{80}{16} = 5$$

So $L(v) - 200 = 5(v-80) \Rightarrow L(v) = 5v - 120$

So, $L(81) = 5(81) - 120 = 328 - 120 = 208$

Thus, $\boxed{R\Big|_{v=81} \approx 208 \text{ ft.}}$

$\boxed{4.2.46}$ Range $R = \frac{1}{32} v^2 \sin(2\theta)$, $v = 80$ ft/sec fixed,
θ inc. from $60°$ to $61°$.

Well, $R = \frac{1}{32}(80^2)\sin(2\theta) = 200\sin(2\theta)$; $R\Big|_{\theta=60°} = 200\sin(2\cdot 60°) = 200\sin(120°)$
$= 200\left(\frac{\sqrt{3}}{2}\right) = 100\sqrt{3}$

$$\frac{dR}{d\theta} = 200\cos(2\theta)\frac{d}{d\theta}(2\theta) = 400\cos(2\theta)$$

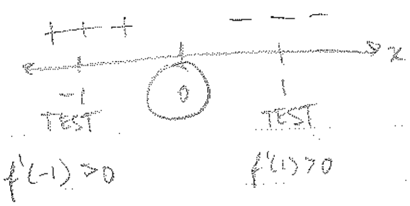
$$\Rightarrow \frac{dR}{d\theta}\Big|_{\theta=60°} = 400\cos(2\cdot 60°) = 400\cdot\cos(120°) = -\frac{1}{2}$$

So, $L(\theta) - 100\sqrt{3} = -\frac{1}{2}(x-60) \Rightarrow L(\theta) = 30 + 100\sqrt{3} - \frac{1}{2}x$

$L(61°) = 30 + 100\sqrt{3} - \frac{61}{2} = 100\sqrt{3} - \frac{1}{2}$.

Thus, $\boxed{R\Big|_{\theta=61} \approx 100\sqrt{3} - \frac{1}{2}.}$

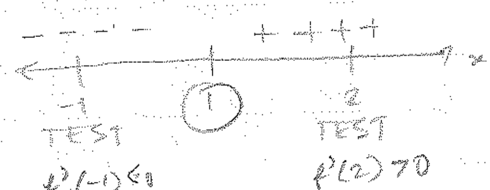
4.3.1 $f(x) = 4 - x^2 \Rightarrow f'(x) = -2x$, and so $f'(x) = 0$ only when $x = 0$, and is always defined. Test:



Inc. on $(-\infty, 0)$
Dec. on $(0, +\infty)$

Must be graph (c).

4.3.2 $f(x) = x^2 - 2x - 1 \Rightarrow f'(x) = 2x - 2 = 2(x-1)$, which is zero only when $x = 1$, and is always defined.



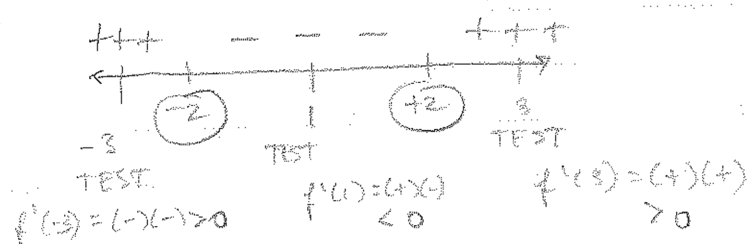
Inc. on $(1, +\infty)$
Dec. on $(-\infty, 1)$

Must be graph (b)

4.3.4 $f(x) = \frac{x^3}{4} - 3x \Rightarrow f'(x) = \frac{3}{4}x^2 - 3 = 3\left(\left(\frac{x}{2}\right)^2 - 1\right) =$

$$= 3\left(\left(\frac{x}{2}\right) + 1\right)\left(\frac{x}{2} - 1\right)$$

And so $f'(x) = 0$ only when $x = -2$ or $x = +2$, and is always defined.



Inc. on $(-\infty, -2) \cup (2, +\infty)$
Dec. on $(-2, +2)$

$\boxed{4.3.8}$ Know $f'(x) = 3\sqrt{x}$, $f(0) = 4$. Find $f(x)$.

Well, we know $\frac{d}{dx}\left[x^{3/2}\right] = \frac{3}{2}x^{1/2}$, so $\frac{d}{dx}\left[2x^{3/2}\right] = 3\sqrt{x}$.

By the corollary to the MVT, then, we must have that $f(x) = 3\sqrt{x} + K$, some $K \in \mathbb{R}$. But we knew $f(0) = 4$, so

$f(0) = 3\sqrt{0} + K = K \Rightarrow K = 4$; finally, $\boxed{f(x) = 3\sqrt{x} + 4.}$

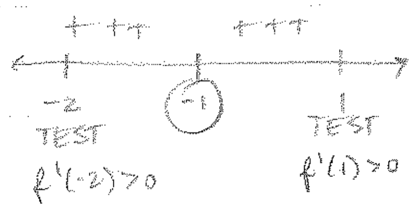
$\boxed{4.3.18}$ Find intervals where inc., dec.: $f(x) = \frac{x}{x+1}$.

Note, $f'(x) = \frac{d}{dx}\left[x(x+1)^{-1}\right] = \frac{d}{dx}[x](x+1)^{-1} + x\frac{d}{dx}\left[(x+1)^{-1}\right]$

$$= \frac{1}{x+1} + \frac{-x}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}.$$

Now, $f'(x)$ is undefined when $(x+1)^2 = 0$, or when $x = -1$.

Test:



So the function f is always increasing—i.e., on $(-\infty, +\infty)$ and is never decreasing.

$\boxed{4.3.26}$ Show tht. f satisfies hypotheses of Rolle's theorem, and find all $x \in (a,b)$ tht. satisfy the conclusion of Rolle's thm. on that interval.

$f(x) = 9x^2 - x^4$ on $[-3,3]$.

Need to show: ① f is cts. on $[-3,3]$
② f is diff'ble on $(-3,3)$
③ $f(-3) = 0 = f(3)$

Now, observe tht. f is diff'ble everywhere on $\mathbb{R}$ (its derivative is $f'(x) = 18x - 4x^3$), and so is cts. on all $\mathbb{R}$ as well, including on $[-3,3]$.

Also, $f(-3) = 9(-3)^2 - (-3)^4 = 81 - 81 = 0$
$f(3) = 9(3)^2 - (3)^4 = 81 - 81 = 0$
so Rolle's thm. holds on $[-3,3]$, and we are guaranteed the existence of some $c \in (-3,3)$ s.t. $f'(c) = 0$.

For us, this c satisfies $18c - 4c^3 = 0 \iff 2c(9 - 2c^2) = 0$.

Thus, $c = 0$, or $c = \pm\sqrt{4.5}$; all of these are in $(-3,3)$, and all satisfy the conclusion of Rolle's thm.

$\boxed{4.3.30}$ $f(x) = 1 - (2-x)^{2/3}$, $[1,3]$ is not diff'ble at $x=2$, so doesn't satisfy the hypotheses of Rolle's thm.

Consider $f'(x) = +\frac{2}{3}(2-x)^{-1/3}$, which is never 0 on $[1,3]$, so the conclusion of Rolle's thm. doesn't hold here.

$\boxed{4.3.34}$ $f(x) = \sqrt{x-1} \Rightarrow f'(x) = \frac{1}{2\sqrt{x-1}}$, which exists on $(2,5)$, so f is diff'ble and cts. there; also, is left cts. at $x=5$ and right cts. at $x=2$, so the hypotheses of the MVT are satisfied.

Now, $f(2) = \sqrt{2-1} = \sqrt{1} = 1$
$f(5) = \sqrt{5-1} = \sqrt{4} = 2$, so $\frac{f(5)-f(2)}{5-2} = \frac{2-1}{5-2} = \frac{1}{3}$.

We seek $c \in (2,5)$ s.t. $f'(c) = \frac{1}{3}$, i.e., $\frac{1}{2\sqrt{c-1}} = \frac{1}{3} \Leftrightarrow \sqrt{c-1} = \frac{3}{2} \Leftrightarrow$

$c - 1 = \frac{9}{4} \Leftrightarrow c = \frac{5}{4} = 1.25$.

$\boxed{4.3.44}$ Prove $\sin x = 3x - 1$ has exactly one sol'n in $[-1,1]$.

Well, first apply the IVT to $f(x) = \sin x - 3x + 1$, noting that f is cts. on all $\mathbb{R}$, including on $[-1,1]$. We have $f(-1) = \sin(-1) + 4 > 0$, and $f(1) = \sin(1) - 2 < 0$, so $\exists c \in (-1,1)$ s.t. $f(c) = 0$.

But $f'(x) = -\cos x - 3 < 0$ on $[-1,1]$, and so f is <u>decreasing</u> on $[-1,1]$ and cannot have more than one zero on this interval.