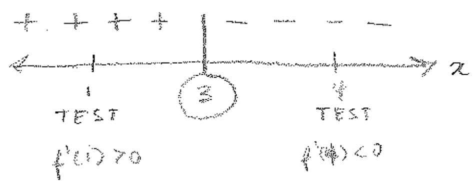
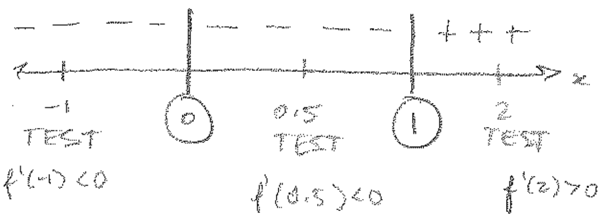
Lecture 9: Example Problems.

4.4.2 $f(x) = 6x - x^2 \Rightarrow f'(x) = 6 - 2x = 2(3-x)$ exists everywhere, and is 0 when $x = 2$. So the only critical point is $x = 3$. Test:



Thus, $x = 3$ is a local maximum of f, and as f decreases on $(3, +\infty)$ and increases on $(-\infty, 3)$, the local max. is a global max.

4.4.12 $f(x) = x^2 + \frac{2}{x} \Rightarrow f'(x) = 2x - \frac{2}{x^2} = 2\frac{x^3 - 1}{x^2}$ is undefined when $x = 0$ and is 0 when $x = 1$. Thus, the critical points are $x = 0$ and $x = 1$. Test:



So, $x = 0$ is not an extremum, and $x = 1$ is a local minimum. But because f decreases on $(-\infty, 1)$ and increases thereafter, the local minimum is actually a global minimum.

4.4.34 Find the point on the parabola $y = 4 - x^2$ closest to the point $(3,4)$.

Well, a point on the parabola is $(x, 4-x^2)$, and the distance btwn. this point and $(3,4)$ is $\sqrt{(x-3)^2 + (4-x^2-4)^2} = \sqrt{x^2 - 6x + 9 + x^4}$, which will be minimized when $f(x) := x^4 + x^2 - 6x + 9$ is minimized.

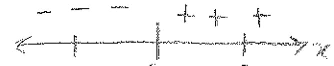
$f'(x) = 4x^3 + 2x - 6$, which is never undefined, and which has $x = 1$ as one of its roots (this is true when the sum of the coefficients is zero), so $(x-1)$ can be factored out:

$$\begin{array}{r} 4x^2 + 4x + 6 \\ (x-1)\overline{\smash{)}\,4x^3 + 2x - 6} \\ -4x^3 + 4x^2 \\ \hline 4x^2 + 2x - 6 \\ -4x^2 + 4x \\ \hline 6x - 6 \end{array}$$

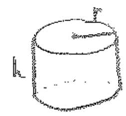
$\Rightarrow f'(x) = (x-1)(4x^2 + 4x + 6) = 2(x-1)(2x^2 + 2x + 3)$,

which is zero when $x = 1$ or when $(2x^2 + 2x + 3) = 0$, which is never. The only c.p. of f is thus $x = 1$, which we really hope is a global minimum. Test:

So, it is a local min and as f dec. on $(-\infty, 1)$ and inc. thereafter, it is a global min, too.

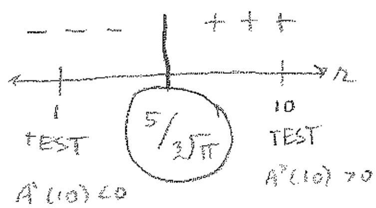


$\boxed{4.4.32}$



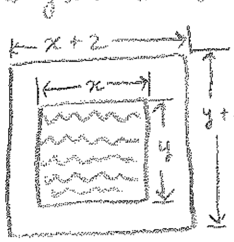
An open-topped cylinder needs 125 cm^3 volume. What dimensions minimize surface area? (Assume negligible wall thickness)

Well, $V = \pi r^2 h$, and $A_{total} = A_{bottom} + A_{wall} = \pi r^2 + 2\pi r h$. Thus, we need $\pi r^2 h = 125 \Rightarrow h = \frac{125}{\pi r^2}$, so $A = \pi r^2 + \frac{2\pi r(125)}{\pi r^2} = \frac{\pi r^3 + 250}{r}$, and $A'(r) = \frac{r(3\pi r^2) - (\pi r^3 + 250)1}{r^2} = \frac{3\pi r^3 - \pi r^3 - 250}{r^2} = \frac{2\pi r^3 - 250}{r^2}$, which is not defined when $r = 0$ (but we should exclude this from the domain anyway), and which is 0 when $r^3 = \frac{250}{2\pi} = \frac{125}{\pi} \Rightarrow r = \frac{5}{\sqrt[3]{\pi}}$. Now, test:



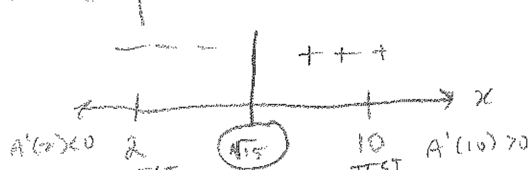
Thus, $r = \frac{5}{\sqrt[3]{\pi}}$ is a local min, and as f decreases on $(-\infty, 5/\sqrt[3]{\pi})$ and increases thereafter, it is also a global min. So, $r = 5/\sqrt[3]{\pi}$ and $h = 125/\pi r^2 = 5/\sqrt[3]{\pi}$.

$\boxed{4.4.40}$ Each page of a book will contain 30 in^2 of print, and each page must have 2-in. margins at top & bottom and 1-in margins at each side. What is the minimum possible area of such a page?



Know $xy = 30$, want to minimize the area function $A(x) = (x+2)(y+4) = \underbrace{xy}_{30} + 4x + \underbrace{2y}_{\frac{60}{x}} + 8 = 4x + \frac{60}{x} + 38$.

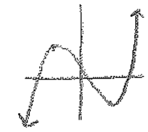
Well, $A'(x) = 4 - \frac{60}{x^2} = \frac{4(x^2 - 15)}{x^2}$ is undefined when $x = 0$ (also, f is undefined there, so 0 was not in domain) and is 0 when $x = \pm\sqrt{15}$. For our problem, $x < 0$ doesn't make sense (negative text width?), so the only critical point we care abt. is $x = \sqrt{15}$. Test:



So, $x = \sqrt{15}$ and $y = 2\sqrt{15}$ solve the problem (x is a global min, because A dec. on $(-\infty, \sqrt{15})$ and inc. on $(\sqrt{15}, \infty)$).

4.5.1 $f(x) = x^3 - 5x + 2 \Rightarrow f'(x) = 3x^2 - 5$ is zero when $x = \pm\sqrt{\frac{5}{3}}$:

Inc. on $(-\infty, -5/3) \cup (5/3, +\infty)$. That is: (c)

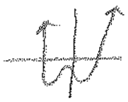


COULD DO THIS – OR...

4.5.2 $f(x) = x^4 - 3x^2 + x - 2$ has $\lim_{|x| \to \infty} f(x) = +\infty$. That's only graph (a)

POSITIVE LEADING COEFF

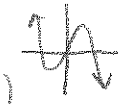
EVEN HIGHEST POWER



4.5.3 $f(x) = -\frac{1}{3}x^5 - 3x^2 + 3x + 2$ has $\lim_{x \to +\infty} f(x) = -\infty$ and $\lim_{x \to -\infty} f(x) = +\infty$: (d)

NEG. LEADING COEFF

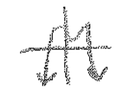
ODD HIGH POWER



4.5.4 $f(x) = -\frac{1}{3}x^6 + 2x^5 - 3x^4 + \frac{1}{2}x + 5$ has $\lim_{|x| \to \infty} f(x) = -\infty$. Only graph (b):

NEG LEAD COEFF

EVEN HIGH POWER



4.5.6 $f(x) = 27 + 12x - 4x^2 \Rightarrow f'(x) = -8x + 12 = 4(-2x + 3)$, which is defined everywhere and which is 0 when $x = 3/2$. The only c.p. is at $x = 3/2$, and we can see from the graph that f is increasing on $(-\infty, 3/2)$ and decreasing on $(3/2, +\infty)$.

4.5.14 $y = 3x^8 - 52x^6 + 216x^4 - 500 \Rightarrow$

$$\frac{dy}{dx} = 24x^7 - 6(52)x^5 + 4(216)x^3$$
$$= 24x^3(x^4 - 13x^2 + 36)$$
$$= 24x^3(x^2 - 9)(x^2 - 4)$$
$$= 24x^3(x+3)(x-3)(x+2)(x-2),$$

which is always defined and which is 0 when $x = 0, \pm 2, \pm 3$.

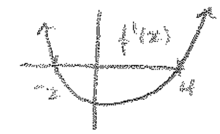
From the graph:

f is decreasing on $(-\infty, -3) \cup (-2, 0) \cup (2, 3)$

increasing on $(-3, -2) \cup (0, 2) \cup (3, +\infty)$.

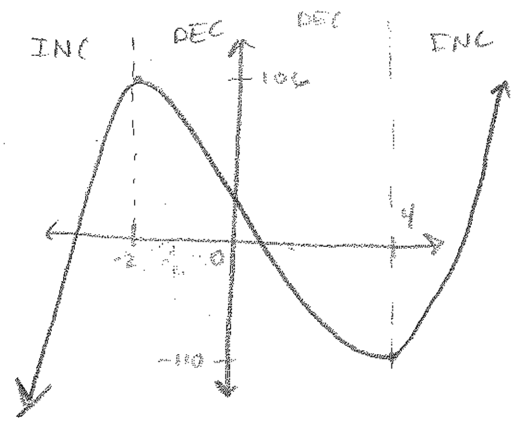
Lec 9 probs, ct'd.

$\boxed{4.5.50}$ $f(-2)=106$, $f(4)=-110$ are critical pts, deriv.:

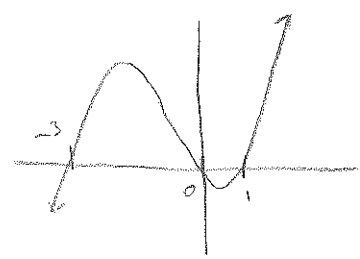


So, f is inc. on $(-\infty,-2)\cup(4,+\infty)$

dec. on $(-2,4)$



$\boxed{4.5.52}$ $f(-3)=-130$, $f(0)=5$, $f(1)=-2$, deriv:



So, f inc. on $(-3,0)\cup(1,+\infty)$

dec. on $(-\infty,-3)\cup(0,1)$