Calculus I
E1 Term, Sections E101 and E196
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Week 2: Solutions to Written Homework Problems

Problem 1. Recall the "epsilon-delta" definition of the limit:

> Suppose that $f(x)$ is defined in an open interval containing the point a (except possibly not at a itself). Then we say that the number L is the *limit of* $f(x)$ *as* x *approaches* a—and we write
>
> $$\lim_{x \to a} f(x) = L$$
>
> —provided that the following criterion is satisfied: Given any number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that
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> $$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

(a) Use this definition to prove that $\lim_{x \to 3} 3x + 7 = 16$.

Solution. Let $\varepsilon > 0$ be given, and suppose that $\delta < \frac{\varepsilon}{3}$. Then the following chain of logical statements holds:

$$\begin{aligned} 0 < |x-3| < \delta &\implies |x-3| < \frac{\varepsilon}{3} \\ &\implies 3\,|x-3| < \varepsilon \\ &\implies |3x-9| < \varepsilon \\ &\implies |(3x+7) - 16| < \varepsilon \\ &\implies |f(x) - 16| < \varepsilon. \end{aligned}$$

Notice, in particular, that $0 < |x - 3| < \delta$ implies that $|f(x) - 16| < \varepsilon$, and this is just what we needed.

This is an extremely important exercise to see how to play the "epsilon-delta game"; that is, someone has given you an $\varepsilon > 0$, and it is your job to come up with some $\delta > 0$ (usually dependent upon ε) that makes the implication $0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$ always true. These proofs always take the following form:

- Let $\varepsilon > 0$ be given (note that you are *not* assuming a particular value of ε here, and this is why the proof works: because you're conducting the proof for a general, unspecified, arbitrary ε, the result is that no matter *which* numerical value of $\varepsilon > 0$ someone were to actually throw at you, you could come back with a δ that would make the rest of the proof work);
- Suppose that $\delta =$[some expression] or that $\delta <$[some expression] (recall from the notes that as soon as you find one δ that works, then any other value δ' such that $0 < \delta' < \delta$ will also work: this is why it doesn't matter whether you say $\delta =$ or $\delta <$[some expression]);
- Prove that with this choice of δ, the "key implication"

  $$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

  is true.

For some functions—like this one—this process is not very difficult; for others, these "epsilon-delta" proofs are more difficult, which is why it is convenient to have a set of rules for evaluating limits.

(b) How would you **negate** the epsilon-delta definition of the limit? That is, how would you go about proving that, for some given f, a, and L, $\lim_{x\to a} f(x) \neq L$, using epsilons and deltas?

Solution. To prove that $\lim_{x\to a} f(x) \neq L$, it is necessary to find some $\varepsilon > 0$ such that for all $\delta > 0$, there exists some x such that both $0 < |x - a| < \delta$, and $|f(x) - L| \geq \varepsilon$.

To see why this is sufficient to prove that $\lim_{x\to a} f(x) \neq L$, it is important to have fully understood how the proof "game" in part (a) worked; that is, you showed that whichever $\varepsilon > 0$ your opponent threw at you, you were able to come back with some δ that would make the key implication true.

So, in order to defeat you at the game, all that your opponent needs to do is throw you *one single* ε for which you *can't* come up with a δ that would make the rest of the proof work. For that particular ε that your opponent uses to defeat you, then, it must be true that for *every* possible δ that you could have thrown back, the key implication is false.

Note that any implication $a \implies b$ can be shown to be false exactly when a holds and b doesn't (for example, the statement "if you are from Canada, then you are left-handed" is obviously false, and we can prove that it is false by showing a Canadian who is not left-handed). In particular, for our key implication above, we can show that it is false by finding some x for which both $0 < |x - a| < \delta$, and $|f(x) - L| \geq \varepsilon$.

Putting the previous two big ideas together will give you the entire negation as written in the first paragraph.

(c) Formulate precise "epsilon-delta" definitions of the one-sided limits (that is, formulate one definition for the left-hand limit, and one for the right-hand limit).

Solution. For the left-hand limit, we write $\lim_{x\to a^-} f(x) = L$ if given any number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\text{if } a - \delta < x < a, \text{ then } |f(x) - L| < \varepsilon.$$

Similarly, for the right-hand limit, we write $\lim_{x\to a^+} f(x) = L$ if given any number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\text{if } a < x < a + \delta, \text{ then } |f(x) - L| < \varepsilon.$$

Notice that the only things here that are different from the definition in the problem statement are in how we mathematically interpret the first part of the key implication, "if x is within δ of a". For the case when x could be on either side of a, then we would write $a - \delta < x < a + \delta$ (which can be rewritten $-\delta < x - a < \delta$, or most succinctly as $0 < |x - a| < \delta$, which excludes $x = a$ as well). But for the case when x must be either to the left or to the right of δ, respectively, then we would write $a - \delta < x < a$ and $a < x < a + \delta$.

(d) Formulate a precise "M-delta" definition of the infinite limit $\lim_{x\to a} f(x) = +\infty$. Your definition should involve the inequality $f(x) > M$.

Solution. We say that $\lim_{x\to a} f(x) = +\infty$ when, for all $M > 0$, there exists $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } f(x) > M.$$

Problem 2. **(a)** Prove, using the definition of continuity, that the function $f(x) = 6^x$ is continuous everywhere on the real line.

Solution. It will be sufficient to show that for all $a \in \mathbb{R}$, $\lim_{x\to a} 6^x = 6^a$. But we had no handy laws for the limits of exponentials, so we need to use the epsilon-delta definition to prove this. I will first describe the line of reasoning that should have led you to the correct epsilon-delta proof, and then I'll give the proof itself.

So, our proof should start off by letting $\varepsilon > 0$ be given, and we should find some δ such that if $|x - a| < \delta$, then $|6^x - 6^a| < \varepsilon$. Let's look at the ε inequality, to see if we can't get it into a form that makes the necessary restriction on δ more apparent.

$$|6^x - 6^a| = 6^a \left|6^{x-a} - 1\right|, \text{ since for all } a \in \mathbb{R},\ 6^a > 0. \text{ Thus,}$$

$$|6^x - 6^a| < \varepsilon \iff 6^a \left|6^{x-a} - 1\right| < \varepsilon \iff \left|6^{x-a} - 1\right| < \frac{\varepsilon}{6^a} \iff -\frac{\varepsilon}{6^a} < 6^{x-a} - 1 < \frac{\varepsilon}{6^a}.$$

Now, let's look at the δ inequality; recall that $|x - a| < \delta$ means that $-\delta < x - a < \delta$, and this means:

$$6^{-\delta} < 6^{x-a} < 6^{\delta} \iff 6^{-\delta} - 1 < 6^{x-a} - 1 < 6^{\delta} - 1.$$

So, if we can find a $\delta > 0$ such that *both*

$$6^{\delta} - 1 < \frac{\varepsilon}{6^a} \quad \text{and} \quad 6^{-\delta} > -\frac{\varepsilon}{6^a},$$

then we will have satisfied the ε inequality. Let's look at the positive side first:

$$6^{\delta} - 1 < \frac{\varepsilon}{6^a} \iff 6^{\delta} < \frac{\varepsilon}{6^a} + 1 \iff \delta < \log_6\left(\frac{\varepsilon}{6^a} + 1\right).$$

So, we need that $\delta < \log_6\left(\frac{\varepsilon}{6^a} + 1\right)$. We also need the negative side; that is:

$$6^{-\delta} - 1 > -\frac{\varepsilon}{6^a} \iff 6^{-\delta} > 1 - \frac{\varepsilon}{6^a} \iff -\delta > \log_6\left(1 - \frac{\varepsilon}{6^a}\right).$$

So, we need that $\delta < -\log_6\left(1 - \frac{\varepsilon}{6^a}\right)$. The result is that if we define $\delta_1 := \log_6\left(\frac{\varepsilon}{6^a} + 1\right)$ and $\delta_2 := -\log_6\left(1 - \frac{\varepsilon}{6^a}\right)$, then the ε inequality will hold if we let δ be smaller than either δ_1 or δ_2; that is, if we let $\delta < \min\{\delta_1, \delta_2\}$.

That was the outline of the reasoning of the proof. The proof itself follows:

Let $\varepsilon > 0$ be given, and let $\delta_1 := \log_6\left(\frac{\varepsilon}{6^a} + 1\right)$ and $\delta_2 := -\log_6\left(1 - \frac{\varepsilon}{6^a}\right)$. Let $\delta < \min\{\delta_1, \delta_2\}$. Suppose that $0 < |x - a| < \delta$. Then we have $x - a < \delta$, and $\delta < \delta_1$ implies

$$6^{x-a} - 1 < 6^{\delta} - 1 < 6^{\delta_1} - 1 = \left(\frac{\varepsilon}{6^a} + 1\right) - 1 = \frac{\varepsilon}{6^a},$$

which implies that $6^{x-a} - 1 < \frac{\varepsilon}{6^a}$, and multiplying through by 6^a, we see that $6^x - 6^a < \varepsilon$.

But also, we have $x - a > -\delta$, and $\delta < \delta_2$ implies that $-\delta > -\delta_2$, which implies

$$6^{x-a} - 1 > 6^{-\delta} - 1 > 6^{-\delta_2} - 1 = \left(1 - \frac{\varepsilon}{6^a}\right) - 1 = -\frac{\varepsilon}{6^a},$$

which implies that $6^{x-a} - 1 > -\frac{\varepsilon}{6^a}$, and multiplying through by 6^a, we see that $6^x - 6^a > -\varepsilon$. Thus, we see that $|6^x - 6^a| < \varepsilon$, and so we have proven that $\lim_{x\to a} 6^x = 6^a$.

I did not expect each of you to construct this epsilon-delta proof perfectly, but I did expect you to realize that since we had not learned any limit laws about exponentials, then we needed to construct some kind of a proof like this. You'll get full credit if you tried this kind of a proof, and didn't just make something up or write nonsense.

(b) Prove, using the Intermediate Value Theorem, that there is a positive, real solution of the equation $x^3 + 3 = 6^x$.

> **Solution.** Fortunately, this is the easier part of this problem, once we establish the continuity of 6^x. In order to use the Intermediate Value Theorem, we need a function and a closed interval on which that function is continuous. For this problem, we should define the function $f(x) := x^3+3-6^x$, and we would like to find some closed interval where f is continuous, where the function values at the endpoints of the interval have opposite sign (we need this to be true because our goal was to show that for some c inside the interval, $f(c) = 0$). Our function f is continuous over any interval of real numbers, though: we had a rule that told us all polynomials were continuous over the entire real line, and $x^3 + 3$ is a polynomial, so it is continuous on $\mathbb{R}$; we also know from part (a) that 6^x is continuous on $\mathbb{R}$, and we also had a rule that the difference of two continuous functions is also continuous. Thus, $f(x)$ is continuous over all of $\mathbb{R}$.
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> Now, we find a suitable interval. The easiest way to do this is by testing points; you might luckily stumble upon the interval $[0, 1]$ for your problem, and in this case, $f(0) = 0^3 + 3 - 6^0 = 3 - 1 = 2$, whereas $f(1) = 1^3 + 3 - 6^1 = -2$. So, the function values at the endpoints have opposite sign, and we've found a good interval to apply the IVT on.
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> Now, we put all of that together. Since $K = 0$ is between $f(0) = 2$ and $f(1) = -2$, and because $f(x) = x^3 + 3 - 6^x$ was continuous over the closed interval $[0, 1]$, we apply the Intermediate Value Theorem to get the result that there must be some $c \in (0, 1)$ such that $f(c) = 0$. Thus, $x = c$ is a solution of the equation $x^3 + 3 = 6^x$.

Problem 3. (a) Establish, using the definition of the derivative, that the derivative of $f(x) = \frac{c}{x}$ is $f'(x) = -\frac{c}{x^2}$, if c is a constant.

> **Solution.** The derivative is given by its definition as:
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> $$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\frac{c}{x+\Delta x} - \frac{c}{x}}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{cx - c(x+\Delta x)}{x\Delta x(x+\Delta x)} \\ &= \lim_{\Delta x\to 0} \frac{-c\Delta x}{x\Delta x(x+\Delta x)} \\ &= \lim_{\Delta x\to 0} \frac{-c}{x^2 + x\Delta x}, \end{aligned}$$
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> and we can evaluate the latter limit by substitution to obtain, finally,
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> $$f'(x) = \lim_{\Delta x\to 0} \frac{-c}{x^2 + x\Delta x} = -\frac{c}{x^2},$$
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> as desired.

(b) The volume V (in liters) of 3 g of CO_2 at 27°C is given n terms of its pressure p (in atmospheres) by the formula

$$V = \frac{1.68}{p}.$$

What is the rate of change of V with respect to p when $p = 2$ atm?

> **Solution.** This problem asked for an instantaneous rate of change, at the instant when $p = 2$ atm. That is, it asked for the derivative of V with respect to p, evaluated at $p = 2$. We compute this using the result in part (a), taking $c = 1.68$:
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> $$V'(2) = \left.\frac{\mathrm{d}V}{\mathrm{d}p}\right|_{p=2} = \left.\frac{-1.68}{p^2}\right|_{p=2} = \frac{-1.68}{4} = 0.42\frac{\text{L}}{\text{atm}}.$$

(c) Plot $V(p)$ and $V'(p)$ on the same graph. Be sure to include all appropriate axis labels, arrows, and scale markings.

Solution. Please see the plot below, where $V(p)$ is marked in blue, and $V'(p)$ is marked in red. Note that $V'(p)$ is always negative, because $V(p)$ is always decreasing, and note that there are no local maxima or minima or $V(p)$, so $V'(p)$ does not cross the p-axis.