Calculus I
E1 Term, Sections E101 and E196
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Week 3: Homework Solutions

Problem 1. Prove, using the definition of the derivative along with a trigonometric sum-to-product formula and some pre-established trigonometric limits, that $\frac{\mathrm{d}}{\mathrm{d}x}[\cos(x)] = -\sin(x)$. For this problem, I will be strictly enforcing the rule about using complete sentences, so I do **not** want to see just a string of equalities with no context. Please consult the similar proof on Page 170 of the text for an example of the type of proof you will be expected to write for this. Borrow the phrasing of the textbook if you must, or try to improve upon it if you can, modifying the argument to fit this problem.

Solution. To differentiate $f(x) = \cos x$, we begin with the definition of the derivative:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}.$$

Next, we apply the addition formula for the cosine:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

together with the limit law for linear combinations:

$$\lim_{x\to c} [af(x) + bg(x)] = a\left[\lim_{x\to c} f(x)\right] + b\left[\lim_{x \to c} g(x)\right],$$

to obtain:

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(\cos x \cos \Delta x - \sin x \sin \Delta x) - \cos x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[-(\cos x)\frac{1 - \cos \Delta x}{\Delta x} - (\sin x)\frac{\sin \Delta x}{\Delta x}\right] \\ &= -(\cos x)\left[\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x - 1}{\Delta x}\right] - (\sin x)\left[\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}\right]. \end{aligned}$$

The values of the above limits are known to be

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0,$$

and substituting these values into our derivative, we obtain

$$f'(x) = -(\cos x)(0) - (\sin x)(1) = -\sin x,$$

just as desired.

At least one of you took the problem statement to mean that I didn't want to see any equations at all in your writeup, and wrote out each equation just as it would be spoken using words. Though I appreciate the effort, that was not at all necessary: the best proofs are complete but concise, and part of being concise means using the symbols and notation shortcuts that you have available to you. What I meant by the problem statement was more about completeness than about conciseness, because a good rule of thumb about proofs in homework assignments is that they should be written for an audience of students one year behind yourselves—given this rule, then, a complete proof that $\frac{\mathrm{d}}{\mathrm{d}x}[\cos(x)] = -\sin(x)$ really needs to explain its steps, and the only way to do that is by using sentences.

So, I'm sorry if the problem statement mislead a few of you into doing more work than was necessary—but most of you borrowed the book's phrasing and got full credit.

Problem 2. The Robbins cinquefoil (*Potentilla robbinsiana*) is a small, extremely rare, yellow-flowered alpine plant found exclusively in New Hampshire's White Mountains. It was on the list of endangered species from the 1980s until 2002, when recovery efforts became successful[1].

(a) At the time the Robbins cinquefoil was taken off the endangered species list, there were about 14,000 plants in a single group on Mount Washington. If we assume that the growth of the plant population is exponential and that the population of this group will double every six years, then a formula for the number of plants at time t, where t represents the number of years after June 2002, is

$$N(t) = 14,000 \cdot 2^{t/6}.$$

How many plants do we expect to be in this group today, exactly twelve years after June 2002?

Solution. This problem asks us to calculate $N(12)$ to determine that $N(12) = 14,000 \cdot 2^{12/6} = 14,000 \cdot 2^2 = 14,000 \cdot 4 = 56,000$ plants should be in the group today.

(b) When do we expect 100,000 plants to be in this group? (Please give a year and a month.)

Solution. First, we find t such that $N(t) = 100,000$; that is,

$$\begin{aligned} N(t) = 100,000 &\iff 14,000 \cdot 2^{t/6} = 100,000 \\ &\iff 2^{t/6} = \frac{100,000}{14,000} \\ &\iff \frac{t}{6} = \log_2\left(\frac{100,000}{14,000}\right) \\ &\iff t = 6\left[\log_2(100,000) - \log_2(14,000)\right] \approx 17.0190 \text{ years.} \end{aligned}$$

Writing t in years and months, we see that the whole-number portion of t is the number of years, and the decimal portion will give the number of months as (# months)= $0.0190 \cdot (12 \text{ months}) = 0.2280$ months. Since we were asked just for a year and a month, we should round this number to the nearest whole number to obtain the number of months to add to the 17 years we found above. Since this particular number rounds to zero, we do not need to add any months to our answer, and we finally find that 100,000 plants are expected to be in the group 17 years and 0 months after June 2002—that is, in June 2019[a].

[a] If we'd wanted to be more thorough, we could have used a similar procedure to see how many days were in 0.2280 of the month of July 2019 (the month after June 2019). We obtain $0.228 \cdot 31$ days ≈ 7 days—so, $N = 100,000$ on 07 July 2019.

(c) How fast is the number of plants expected to grow now (twelve years after 2002), and how fast is it expected to grow two years from now (fourteen years after 2002)? Please give your answer in units of plants per year.[2]

Solution. Since this problem asks for two instantaneous rates of change, we must calculate the derivative of $N(t)$, which we do as follows, using the constant multiplication rule, the chain rule, and the rule for differentiating exponential functions:

$$N'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[14,000 \cdot 2^{t/6}\right] = 14,000 \frac{\mathrm{d}}{\mathrm{d}t}\left[2^{t/6}\right] = 14,000 \ln(2) 2^{t/6} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{t}{6}\right] = \frac{14,000\ln(2)}{6} 2^{t/6}.$$

We now compute the desired quantities; that is, $N'(12)$ and $N'(14)$, to find that today, the number of plants in the group is expected to grow at

$$N'(12) = \frac{14,000\ln(2)}{6} 2^{12/6} = \frac{14,000\ln(2) \cdot 2^2}{6} = \frac{56,000\ln(2)}{3} \approx 6,469 \text{ plants per year,}$$

and two years from today, the number of plants in the group is expected to grow at

$$N'(14) = \frac{14,000\ln(2)}{6} 2^{14/6} = \frac{14,000\ln(2) \cdot 2^{7/3}}{6} \approx 8,151 \text{ plants per year.}$$

[1] The information about this peculiar little plant was taken from the following site, where you can read more about its ecology: http://www.centerforplantconservation.org/collection/cpc_viewprofile.asp?CPCNum=3609

[2] Hint: this problem asks for the instantaneous rates of change of the function.

Problem 3. You have decided to abandon your science education and go into business selling vegetarian burritos out of a food truck in Manhattan. Suppose that your fixed costs are \$500 per day, and that each burrito costs you \$1.00 to make.

(a) Let b be the number of burritos that you sell per day, and write your costs per day as a function of b.

Solution. The daily costs will be $C(b) = 500 + b$ dollars.

(b) If you charge a price of p dollars per burrito, then write your gross daily income as a function of b (supposing that p is constant), and write your net daily profit as a function of the variable b (your net profit is the gross income, minus the total costs). Call the net daily profit function $M(b)$, and note that your expression for $M(b)$ should involve p, which for now, we are supposing is constant.

Solution. The gross daily income will be $I(b) = pb$ dollars, and the net daily profit will be $M(b) = I(b) - C(b) = pb - (500 + b) = (p - 1)b - 500$ dollars.

(c) Of course, the more money you charge for your burritos, the fewer people will want to buy them. Suppose that you've been winging it for the first few weeks, but using this time to record valuable sales data, and you've found that if you control for all other factors, you will sell 150 burritos if you charge \$7.00 per burrito, and you will sell 300 burritos if you charge \$5.00 per burrito[3]. Write b as a linear function of p (I suggest you simply use the point-slope formula).

Solution. The equation of the line in (p, b) containing the two points $(7, 150)$ and $(5, 300)$ is

$$b - 300 = \frac{300 - 150}{7 - 5}(p - 5) = -75(p - 5) = -75p + 375 \iff b = 675 - 75p.$$

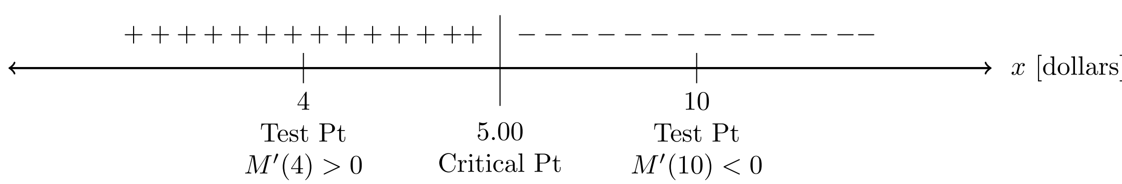
(d) Rewrite the net daily profit M that you found in part (b) as a function of p only, using your result from part (c). Your answer will be a function $M(p)$.

Solution. Substituting the value of b into M yields $M(p) = (p - 1)(675 - 75p) - 500 = (675p - 75p^2 - 675 + 75p) - 500 = -75p^2 + 750p - 1175$.

(e) How much should you charge per burrito in order to maximize your daily profits? What will your maximum daily profit be if you charge this amount?

Solution. This problem asks you to optimize $M(p)$. If we frame this as a global optimization problem, then we should differentiate, find the critical points, check the sign of $M'(p)$ around the critical points, and check the limiting behavior of the function.

So, $M'(p) = \frac{\mathrm{d}}{\mathrm{d}t}\left[-75p^2 + 750p - 1175\right] = -75(2)p + 750 = 750 - 150p$, and this is always defined, so the only critical points will occur when $M'(p) = 0$; that is, when $p = \frac{750}{150} = \$5.00$. We test the sign of $M'(p)$ on either side of this single critical point:



We see that $p \approx \$5.00$ is indeed a local maximum; to see that this local maximum is, in fact, the global maximum, we check the limiting behavior:

$$\lim_{p \to -\infty} M(p) = -\infty, \quad \text{and} \quad \lim_{p \to +\infty} M(p) = -\infty,$$

so that at no point could the function value exceed $M(\$5.00)$ =700. Thus, you should charge \$5.00 for each burrito, and you should expect to earn a daily profit of \$700 if you charge this amount.

[3]This is actually an unrealistically high number of sales per day. Figure that it takes you one minute to make a burrito—then you'll be working for five hours *straight* just to make all of the burritos you sell at \$5.00 each. Hope you had an assistant!—This is why you probably don't really want to abandon your science education to make burritos.

Solution. Another way to do this problem would have been to frame it as a local optimization problem, coming up with a closed interval of prices you were willing to charge. The lower endpoint of this interval is easy—make it 0, because it doesn't make sense to charge negative prices. The upper endpoint isn't so obvious. Ideally, you would set the price so that your profit is always non-negative, but without graphing the profit function, or at least realizing that it is a quadratic equation and computing its roots using the quadratic formula or factoring, you wouldn't know exactly when the profit becomes negative. I suggest doing the naïve thing and choosing a really high number—like \$50—that you couldn't imagine would be a realistic price for a burrito, and going from there.

So, the interval we chose for our local optimization problem was $[0, 50]$. To optimize, we'd find the derivative, compute the critical points, and then test the function both at the endpoints, and at whichever of those critical points lie within the chosen interval.

We already found the derivative above, and the corresponding critical point of \$5.67, which does happen to be inside our interval. So we test the function values at these points, and find:

$$\begin{aligned} M(0) &= -75(0)^2 + 750(0) - 1175 = -\$1175 \\ M(5.00) &= -75(5)^2 + 850(5) - 1175 = \$700 \\ M(50) &= -75(50)^2 + 750(50) - 1175 = -\$151,175. \end{aligned}$$

From looking at these points, it's clear that the maximum of $M(p)$ occurs at $(\$5.00, \$700)$, so you should charge \$5.00 for your burritos, and you should expect a daily profit of \$700.

Solution. Because the profit function $M(t)$ is a quadratic polynomial, this problem could also have been done the same way that the oil field production problem was done back in Homework 1. For that assignment, you completed the square to find the coordinates of the vertex of the parabola, then reasoned that since the leading coefficient (we called it a) was negative, the parabola pointed downward and so its maximum would be located at its vertex. That would have worked perfectly fine here—but I'd hoped that you would take your shiny, new derivative tools out for a spin, and most of you did. So, good job.