Calculus I
E1 Term, Sections E101 and E196
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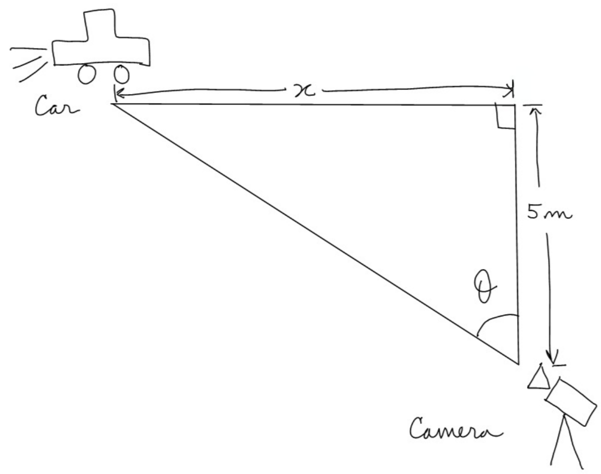
Week 4: Homework Solutions

Problem 1. A camera is located 5 m from a straight road, along which a car is travelling at 50 km/hr. The camera turns so that it is pointed at the car at all times.

(a) [**2 points**] Draw a picture of this scenario, choosing names for and labelling the relevant variables and the fixed quantity that you have been given in the problem statement.



(b) [**2 points**] Suppose that you are asked how fast the camera angle is changing when the car is closest to the camera. (Let $t = T$ represent the time when the car is closest to the camera.) Write down, using the variable $t = T$ and the other variables you chose in part (a), what you are being asked to find (do not do any computations now; just write down what you are looking for). Also, write all of the information that the problem gives you about those variables (again, do not solve for anything). What is the camera angle at time $t = T$?

Solution. You are being asked to find the rate of change of the camera angle, which I called θ, with respect to time t; that is, you are asked to find $\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|_{t=T}$. You are given that the speed of the car—that is, the rate of change of the car's position, which I called x—along the roadway is a constant 50 km/hr, so "in mathematics", this is $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$ km/hr. At the time $t = T$ when the car is closest to the camera, we will have $\theta(T) = 0$.

(c) [**2 points**] What is the relationship between the camera angle and the car's distance from the camera, as measured along the road? Write this out "in mathematics", *i.e.*, using the variables you chose in part (a), and be sure to include units.

Solution. The relationship between the camera angle and the car's distance along the roadway is

$$\tan\theta = \frac{x \text{ m}}{5 \text{ m}}.$$

(d) **[4 points]** Determine how fast the camera angle is changing when the car is closest to the camera, in units of radians per second. This will involve taking appropriate derivatives using implicit differentiation, and converting units before substituting them into your final expression. You are to show the appropriate units at *each* stage of your computation, and your final answer should also show units. Show *ALL* of your work.

Solution. Differentiating both sides of the above relationship, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\tan\theta] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x\ \mathrm{m}}{5\ \mathrm{m}}\right],$$

the left-hand side of which is

$$\frac{\mathrm{d}}{\mathrm{d}t}[\tan\theta] = \sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t},\ \text{which has units of } \frac{1}{\mathrm{sec}},$$

and the right-hand side of which is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x\ \mathrm{m}}{5\ \mathrm{m}}\right] = \frac{1}{5\ \mathrm{m}}\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{m}}{\mathrm{sec}} = \frac{1}{5\ \mathrm{m}}\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{m}}{\mathrm{sec}},\ \text{which also has units of } \frac{1}{\mathrm{sec}}.$$

Thus, we have

$$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{5}\frac{\mathrm{d}x}{\mathrm{d}t},$$

and solving for $\frac{\mathrm{d}\theta}{\mathrm{d}t}$, we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{5\sec^2\theta}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\cos^2\theta}{5}\frac{\mathrm{d}x}{\mathrm{d}t},\ \text{which has units of } \frac{1}{\mathrm{sec}} = \frac{\mathrm{radians}}{\mathrm{sec}},\ \text{since radians are unitless.}$$

We now observe that this formula needs $\frac{\mathrm{d}x}{\mathrm{d}t}$ in terms of meters per second, but we were given units of kilometers per hour. To convert, we use dimensional analysis:

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=T} = \frac{50\ \mathrm{km}}{1\ \mathrm{hr}}\left(\frac{1\ \mathrm{hr}}{60^2\ \mathrm{sec}}\right)\left(\frac{1000\ \mathrm{m}}{1\ \mathrm{km}}\right) = \frac{125}{9}\frac{\mathrm{m}}{\mathrm{sec}}.$$

Now, substituting in the values we know at $t = T$, we finally obtain

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|_{t=T} = \frac{\cos^2\theta(T)}{5}\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=T} = \frac{\cos^2(0)}{5}\left(\frac{125}{9}\right)\frac{\mathrm{radians}}{\mathrm{sec}} = \frac{25}{9}\frac{\mathrm{radians}}{\mathrm{sec}}.$$

Problem 2. This problem will show you how to find roots using Newton's method. See Equation 8 on Page 208 for the square root case.

(a) **[2 points]** Write down the iterative formula for Newton's method, taking $f(x) = x^3 - a$, where a is a constant. Show all of your work and simplify your expression as much as possible.

Solution. If $f(x) = x^3 - a$, then $f'(x) = 3x^2$, and so the iterative formula becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_{n+1})} = x_n - \frac{x_n^3 - a}{3x_n^2} = \frac{3x_n^3 - x_n^3 + a}{3x_n^2} = \frac{2x_n^3 + a}{3x_n^2}.$$

(b) **[2 points]** Write down the iterative formula for Newton's method, taking $f(x) = x^4 - a$, where a is a constant. Again, show all of your work and simplify your expression as much as possible.

Solution. If $f(x) = x^4 - a$, then $f'(x) = 4x^3$, and so the iterative formula becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_{n+1})} = x_n - \frac{x_n^4 - a}{4x_n^3} = \frac{4x_n^4 - x_n^4 + a}{4x_n^3} = \frac{3x_n^4 + a}{4x_n^3}.$$

(c) **[2 points]** Write down the iterative formula for Newton's method, taking $f(x) = x^k - a$, where a and k are both constants. I shouldn't have to tell you that you must show all of your work and simplify your expression as much as possible.

Solution. If $f(x) = x^k - a$, then $f'(x) = kx^{k-1}$, and so the iterative formula becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_{n+1})} = x_n - \frac{x_n^k - a}{kx_n^{k-1}} = \frac{kx_n^k - x_n^k + a}{kx_n^{k-1}} = \frac{(k-1)x_n^k + a}{kx_n^{k-1}}.$$

(d) **[4 points]** Use your formula from part (c) to find $\sqrt[9]{120}$ accurate to five decimal places. Write what you chose $f(x)$ to be, write the resulting iterative formula, write your initial guess x_0, and in general, show *ALL* of your work: you will receive absolutely no credit for giving only the final answer, and your solution must include equations to show how you obtained each of the successive approximations $x_1, x_2, \ldots, x_n$.

Solution. We choose $f(x) = x^9 - 120$, and note that when $f(x) = 0$, we will have $x = \sqrt[9]{120}$. We proceed using Newton's method to find the solution, using the formula from part (c) with $k = 9$ and $a = 120$:

$$x_{n+1} = \frac{8x_n^9 + 120}{9x_n^8}$$

We naïvely use the initial guess $x_0 = 3$, and obtain convergence within nine iterations:

$$\begin{aligned}
x_0 &= 3 \\
x_1 &= \frac{8x_0^9 + 120}{9x_0^8} = \frac{8(3)^9 + 120}{9(3)^8} \approx 2.668698877 \\
x_2 &= \frac{8x_1^9 + 120}{9x_1^8} \approx \frac{8(2.668698877)^9 + 120}{9(2.668698877)^8} \approx 2.377359313 \\
x_3 &= \frac{8x_2^9 + 120}{9x_2^8} \approx \frac{8(2.377359313)^9 + 120}{9(2.377359313)^8} \approx 2.126275406 \\
x_4 &= \frac{8x_3^9 + 120}{9x_3^8} \approx \frac{8(2.126275406)^9 + 120}{9(2.126275406)^8} \approx 1.921936684 \\
x_5 &= \frac{8x_4^9 + 120}{9x_4^8} \approx \frac{8(1.921936684)^9 + 120}{9(1.921936684)^8} \approx 1.780006813 \\
x_6 &= \frac{8x_5^9 + 120}{9x_5^8} \approx \frac{8(1.780006813)^9 + 120}{9(1.780006813)^8} \approx 1.714529977 \\
x_7 &= \frac{8x_6^9 + 120}{9x_6^8} \approx \frac{8(1.714529977)^9 + 120}{9(1.714529977)^8} \approx 1.702584137 \\
x_8 &= \frac{8x_7^9 + 120}{9x_7^8} \approx \frac{8(1.702584137)^9 + 120}{9(1.702584137)^8} \approx 1.702237722 \\
x_9 &= \frac{8x_7^8 + 120}{9x_8^8} \approx \frac{8(1.702237722)^9 + 120}{9(1.702237722)^8} \approx 1.702237440,
\end{aligned}$$

and so, on the ninth iteration, we see that the first five digits after the decimal point have remained unchanged, and we can say that $\sqrt[9]{120} \approx 1.70224$.

The reason why we have shown the calculations to nine digits at each step is that using a greater number of digits of precision helps us avoid what is called "round-off error" in our calculations (actually, these computations were carried out by creating and calling floating-point expressions in MATLAB, so an even greater number of significant digits were used, not only the nine digits shown). It is always good practice to use closed-form solutions (that is, square roots and fractions instead of decimal approximations) whenever possible, and when closed-form expressions are impractical (as they are now), it is good practice to retain as many digits of precision as you can in your calculations, even though the final number of digits needed may be small (five, in this case).

Problem 3. The equation $x + \tan x = 0$ is important in a variety of applications—e.g., in the study of diffusion of heat.

(a) [**1 point**] For which values of x is the function $f(x) = x + \tan x$ continuous?

Solution. Note that as a polynomial, x is continuous on all $\mathbb{R}$, and also, as we have shown in class and in the textbook, $\sin x$ and $\cos x$ are continuous on all $\mathbb{R}$. Thus, by the quotient rule for limits, we will have that $\tan x = \frac{\sin x}{\cos x}$ is continuous wherever it is defined; that is, wherever $\cos x \neq 0$. As a result, $f(x)$ will be continuous wherever $\cos x \neq 0$, that is, where $x \neq \frac{(2n+1)\pi}{2}$, for $n \in \mathbb{Z}$.

(b) [**2 points**] Prove, using the Intermediate Value Theorem, that $x + \tan x = 0$ has at least one solution inside the interval $[1.8, 2.2]$, and at least one solution in the interval $[4.8, 5.2]$.

Solution. We showed in part (a) that f is continuous on the set $\{x \in \mathbb{R} : x \neq \frac{(2n+1)\pi}{2}, \text{ for } n \in \mathbb{Z}\}$, and we should note that the interval $[1.8, 2.2]$ is fully included within this set (since it contains no numbers of the form $(2n+1)\pi/2$). Note also that $f(1.8) \approx -2.49 < 0$, while $f(2.2) \approx 0.83 > 0$, and so because 0 lies between $f(1.8)$ and $f(2.2)$, and because f is continuous on the closed interval $[1.8, 2.2]$, the Intermediate Value Theorem guarantees the existence of at least one $c \in [1.8, 2.2]$ such that $f(c) = 0$, i.e., that $x + \tan c = 0$.

Similarly, $f(4.8) \approx -6.59 < 0$, while $f(5.2) \approx 3.31 > 0$, and so because 0 lies between $f(4.8)$ and $f(5.2)$, and because f is continuous on the closed interval $[4.8, 5.2]$, the Intermediate Value Theorem guarantees the existence of at least one $c \in [4.8, 5.2]$ such that $f(c) = 0$, i.e., that $x + \tan c = 0$.

(c) [**2 points**] Write the iterative formula for Newton's method using $f(x) = x + \tan x$. Show all work and simplify your expression as much as possible. Using this formula may be more convenient if you write it in terms of sine and cosine only.

Solution. If we have

$$f(x) = x + \tan x = x + \frac{\sin x}{\cos x} = \frac{x\cos x + \sin x}{\cos x},$$

then

$$f'(x) = 1 + \sec^2 x = 1 + \frac{1}{\cos^2 x} = \frac{\cos^2 x + 1}{\cos^2 x}.$$

Thus, substituting into Newton's iterative formula, we obtain

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= x_n - \frac{x_n + \tan x_n}{1 + \sec^2 x_n} \\
&= x_n - \frac{x_n \cos x_n + \sin x_n}{\cos x_n} \frac{\cos^2 x_n}{\cos^2 x_n + 1} \\
&= \frac{x_n(\cos^2 x_n + 1) - x_n \cos^2 x_n - \sin x_n \cos x_n}{\cos^2 x_n + 1} \\
&= \frac{x_n - \sin x_n \cos x_n}{\cos^2 x_n + 1}.
\end{aligned}$$

(d) **[2 points]** Using this formula with the initial guess $x_0 = 1.9$, find a numerical approximation of the root within the interval $[1.8, 2.2]$, accurate to six decimal places. Again, you must show *all* of your work in order to receive credit.

Solution. We use the formula to carry out the iterative scheme, resulting in the following sequence:

$$
\begin{aligned}
x_0 &= 1.9 \\
x_1 &= \frac{x_0 - \sin x_0 \cos x_0}{\cos^2 x_0 + 1} = \frac{(1.9) - \sin(1.9)\cos(1.9)}{\cos^2(1.9) + 1} \approx 1.997190314844232 \\
x_2 &= \frac{x_1 - \sin x_1 \cos x_1}{\cos^2 x_1 + 1} = \frac{(1.997190314844232) - \sin(1.997190314844232)\cos(1.997190314844232)}{\cos^2(1.997190314844232) + 1} \\
&\approx 2.027014139539300 \\
x_3 &= \frac{x_2 - \sin x_2 \cos x_2}{\cos^2 x_2 + 1} = \frac{(2.027014139539300) - \sin(2.027014139539300)\cos(2.027014139539300)}{\cos^2(2.027014139539300) + 1} \\
&\approx 2.028752669372769 \\
x_4 &= \frac{x_3 - \sin x_3 \cos x_3}{\cos^2 x_3 + 1} = \frac{(2.028752669372769) - \sin(2.028752669372769)\cos(2.028752669372769)}{\cos^2(2.028752669372769) + 1} \\
&\approx 2.028757838065096 \\
x_5 &= \frac{x_4 - \sin x_4 \cos x_4}{\cos^2 x_4 + 1} = \frac{(2.028757838065096) - \sin(2.028757838065096)\cos(2.028757838065096)}{\cos^2(2.028757838065096) + 1} \\
&\approx 2.028757838110435,
\end{aligned}
$$

and so, on the fifth iteration, we see that the first six digits after the decimal point have remained unchanged, and we can say that the solution to $f(x) = 0$ in the interval $[1.8, 2.2]$ is approximately 2.028758.

(e) **[2 points]** Using this formula with the initial guess $x_0 = 4.9$, find a numerical approximation of the root within the interval $[4.8, 5.2]$, accurate to six decimal places. Again, you must show *all* of your work in order to receive credit.

Solution. We use the formula to carry out the iterative scheme, resulting in the following sequence:

$$
\begin{aligned}
x_0 &= 4.9 \\
x_1 &= \frac{x_0 - \sin x_0 \cos x_0}{\cos^2 x_0 + 1} = \frac{(4.9) - \sin(4.9)\cos(4.9)}{\cos^2(4.9) + 1} \approx 4.912354168503686 \\
x_2 &= \frac{x_1 - \sin x_1 \cos x_1}{\cos^2 x_1 + 1} = \frac{(4.912354168503686) - \sin(4.912354168503686)\cos(4.912354168503686)}{\cos^2(4.912354168503686) + 1} \\
&\approx 4.913177212054434 \\
x_3 &= \frac{x_2 - \sin x_2 \cos x_2}{\cos^2 x_2 + 1} = \frac{(4.913177212054434) - \sin(4.913177212054434)\cos(4.913177212054434)}{\cos^2(4.913177212054434) + 1} \\
&\approx 4.913180439385666 \\
x_4 &= \frac{x_3 - \sin x_3 \cos x_3}{\cos^2 x_3 + 1} = \frac{(4.913180439385666) - \sin(4.913180439385666)\cos(4.913180439385666)}{\cos^2(4.913180439385666) + 1} \\
&\approx 4.913180439434884
\end{aligned}
$$

and so, on the fourth iteration, we see that the first six digits after the decimal point have remained unchanged, and we can say that the solution to $f(x) = 0$ in the interval $[4.8, 5.2]$ is approximately 4.913180.

(f) **[1 point]** Draw a graph of $f(x)$, including all axis labels, tick marks and arrows, noting the scale and clearly marking the positions of all asymptotes. Use the information you've found in the other parts of this problem; you may also wish to use a tool like WolframAlpha to assist you.