Calculus I
E1 Term, Sections E101 and E196
Instructor: E.M. Kiley
June 16, 2014

Week 6: Homework Solutions

Due date: Friday, 27 June 2014, 11:59 p.m. EDT. Please upload a `.pdf` version to myWPI (`my.wpi.edu`).

Problem 1. **(a)** [**5 points**] Prove that the equation $x^7 + x^5 + x^3 + 1 = 0$ has exactly one real solution. Carefully explain your reasoning, and if you refer to a function in your writeup, you must state how you have defined that function. You should be using two big theorems that we learned in class, and in order to do so, you must show explicitly how the hypotheses of each theorem are satisfied.

Solution. There are two parts to this problem: first, proving that the equation has a solution at all, and second, proving that it has only one solution.

To prove the first part, we use the Intermediate Value Theorem. Let $f(x) = x^7 + x^5 + x^3 + 1$, and observe that as $f(x)$ is a polynomial, it is continuous over all $\mathbb{R}$, and in particular, over the interval $[-1, 1]$. Note that $f(-1) = (-1)^7 + (-1)^5 + (-1)^3 + 1 = -2$, and $f(1) = 1^7 + 1^5 + 1^3 + 1 = 4$. Thus, by the Intermediate Value Theorem, since 0 is between $f(-1)$ and $f(1)$, there must be some $c \in (-1, 1)$ such that $f(c) = 0$. Then c solves the equation $x^7 + x^5 + x^3 + 1 = 0$, and we have finished Part (1).

To prove the second part, we Corollary 3 to the Mean Value Theorem, presented and proven on Page 240 of the textbook. Corollary 3 states:

> If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function on $[a, b]$.

So, let's observe that for us, $f'(x) = 7x^6 + 5x^4 + 3x^2$, which is the sum of squares (observe: $f'(x) = (\sqrt{7}x^3)^2 + (\sqrt{5}x^2)^2 + (\sqrt{3}x)^2)$, and so is strictly positive for any $x \neq 0$. Let $a < 0$ be a strictly negative real number, and observe that $f'(x) > 0$ for all $x \in (a, 0)$, so by Corollary 3, f is an increasing function on $[a, 0]$; also, if $b > 0$ is a strictly positive real number, then $f'(x) > 0$ for all $x \in (0, b)$, so by Corollary 3 again, f is an increasing function on $[0, b]$. Thus, since a and b were arbitrarily chosen above, we may say that for all $\mathbb{R}$, f is an increasing function.

However, by the definition of a function that increases over all $\mathbb{R}$, we have that if x_0 and x_1 are real numbers such that $x_0 < x_1$, then $f(x_0) < f(x_1)$. Thus, for our particular function $f(x) = x^7 + x^5 + x^3 + 1$, which we have just shown increases over all $\mathbb{R}$, if $x < c$ (where c is such that $f(c) = 0$), then we must have $f(x) < 0$ and in particular, $f(x) \neq 0$; similarly, if $x > c$, then we must have $f(x) > 0$ and in particular, $f(x) \neq 0$. Thus, there can be only one zero of f in $\mathbb{R}$, and we have proven Part (2).

(b) **[5 points]** Show that the function $f(x) = x^{\frac{2}{3}}$ does *not* satisfy the hypotheses of the Mean Value Theorem over the interval $[-1, 27]$. Show that, nevertheless, there *does* exist a number $c \in (-1, 27)$ such that

$$f'(c) = \frac{f(27) - f(-1)}{27 - (-1)}.$$

Solution. Note that $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$, which increases without bound as $x \to 0$ but does not exist at $x = 0$. Thus, f is not differentiable at the point $x = 0$, which is within the open interval $(-1, 27)$, and so f fails to satisfy this hypothesis of the Mean Value Theorem.

Note that we seek $c \in [-1, 27]$ such that

$$f'(c) = \frac{f(27) - f(-1)}{27 - (-1)} = \frac{9 - 1}{27 + 1} = \frac{8}{28} = \frac{2}{7}.$$

However, observe that if $c = 343/27$, then

$$f'(c) = f'\left(\frac{343}{27}\right) = \frac{2}{3\sqrt[3]{\frac{343}{27}}} = \frac{2}{3\left(\frac{7}{3}\right)} = \frac{2}{7},$$

and moreover, $c \in [-1, 27]$. So, for this choice of $f(x)$, the hypotheses of the MVT did not hold, but we have just shown that the conclusion does hold. **This is okay!** It only shows that the MVT is just a forward implication (an "if-statement"), but it is not a bidirectional implication (an "if-and-only-if statement")[a].

[a] Remember the (true) statement: if an animal is a Bassett Hound, then it is a dog. It is the same kind of statement as the MVT—true in the forward direction, but the converse ("if it is a dog, then it is a Bassett Hound") is false, because a Dalmatian is a dog, but is not a Bassett Hound. The Dalmatian performs the same function for our 'Dog Theorem' that the function $f(x) = x^{2/3}$ does for the MVT: it disproves the converse.

Problem 2. Let $f(x) = x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}$.

(a) **[1 point]** Find the first derivative $f'(x)$ and the second derivative $f''(x)$.

Solution.

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\frac{1}{3}}\right](6-x)^{\frac{2}{3}} + x^{\frac{1}{3}}\frac{\mathrm{d}}{\mathrm{d}x}\left[(6-x)^{\frac{2}{3}}\right] \\
&= \frac{1}{3}x^{-\frac{2}{3}}(6-x)^{\frac{2}{3}} + x^{\frac{1}{3}}(6-x)^{-\frac{1}{3}}\frac{\mathrm{d}}{\mathrm{d}x}[(6-x)] \\
&= \frac{(6-x)^{\frac{2}{3}}}{3x^{\frac{2}{3}}} - \frac{2x^{\frac{1}{3}}}{3(6-x)^{\frac{1}{3}}} \\
&= \frac{(6-x)-2x}{3x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}} \\
&= \frac{2-x}{x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}} \\
f''(x) &= \frac{x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}\frac{\mathrm{d}}{\mathrm{d}x}[2-x] - (2-x)\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}\right]}{x^{\frac{4}{3}}(6-x)^{\frac{2}{3}}} \\
&= \frac{-x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}}{x^{\frac{4}{3}}(6-x)^{\frac{2}{3}}} - \frac{(2-x)}{x^{\frac{4}{3}}(6-x)^{\frac{2}{3}}}\left(\frac{2}{3}x^{-\frac{1}{3}}(6-x)^{\frac{1}{3}} + \frac{1}{3}x^{\frac{2}{3}}(6-x)^{-\frac{2}{3}}\frac{\mathrm{d}}{\mathrm{d}x}[6-x]\right) \\
&= -\frac{1}{x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}} - \frac{2-x}{x^{\frac{4}{3}}(6-x)^{\frac{2}{3}}}\left(\frac{2(6-x)-x}{3x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}\right) \\
&= -\frac{1}{x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}} - \frac{(2-x)(4-x)}{x^{\frac{5}{3}}(6-x)^{\frac{2}{3}}} \\
&= \frac{-x(6-x)-(2-x)(4-x)}{x^{\frac{5}{3}}(6-x)^{\frac{4}{3}}} \\
&= \frac{-6x+x^2-8+6x-x^2}{x^{\frac{5}{3}}(6-x)^{\frac{4}{3}}} \\
&= \frac{-8}{x^{\frac{5}{3}}(6-x)^{\frac{4}{3}}}
\end{aligned}$$

(b) **[3 points]** Find and classify the critical points of f. Where is the tangent line horizontal, where is it vertical, and where does it fail to exist? Evaluate $f(x)$ at each of the critical points. Where is f increasing, and where is it decreasing?

Solution. The critical points of f are those x for which $f'(x) = 0$, or for which $f'(x)$ is undefined. Note that $f'(2) = 0$, so $x = 2$ is a critical point where f has a horizontal tangent line. Also, $f'(0)$ and $f'(6)$ are both undefined, but $f(0)$ and $f(6)$ both exist; thus, $x = 0$ and $x = 6$ are both critical points with vertical tangent lines. Now, let's determine the increasing/decreasing behavior of f, and the local extrema by using the first derivative test:

$+++++++$	|	$+++++++$	|	$-------$	|	$+++++++$	$f'(x)$
-1		1		3		10	x
Test Point	0 Critical Point	Test Point	2 Critical Point	Test Point	6 Critical Point	Test Point	

Where we have evaluated f' at the test points:

$$
\begin{aligned}
f'(-1) &= \frac{2-(-1)}{(-1)^{\frac{2}{3}}(6-(-1))^{\frac{1}{3}}} = \frac{3}{7^{\frac{1}{3}}} > 0 \\
f'(1) &= \frac{2-(1)}{(1)^{\frac{2}{3}}(6-(1))^{\frac{1}{3}}} = \frac{1}{5^{\frac{1}{3}}} > 0 \\
f'(3) &= \frac{2-(3)}{(3)^{\frac{2}{3}}(6-(3))^{\frac{1}{3}}} = -\frac{1}{3^{\frac{2}{3}}3^{\frac{1}{3}}} < 0 \\
f'(10) &= \frac{2-(10)}{(10)^{\frac{2}{3}}(6-(10))^{\frac{1}{3}}} = -\frac{8}{10^{\frac{2}{3}}(-4)^{\frac{1}{3}}} > 0.
\end{aligned}
$$

We thus know that f is increasing on $(-\infty, 2) \cup (6, +\infty)$, and is decreasing on $(2, 6)$. We also know that $x = 0$ is not an extremum, $x = 2$ gives a local maximum, and $x = 6$ gives a local minimum of f. We evaluate f at each of the critical points:

$$f(0) = 0; \qquad f(2) = 2^{\frac{1}{3}}4^{\frac{2}{3}} = \sqrt[3]{32}; \qquad f(6) = 0.$$

Thus, in summary:

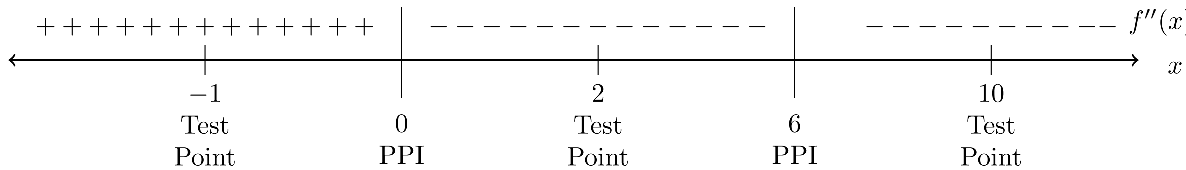
(0,0)	is not an extremum,	has a vertical tangent line
$(2, \sqrt[3]{32})$	is a local maximum,	has a horizontal tangent line
(6,0)	is a local minimum,	has a vertical tangent line

and

f is increasing on $(-\infty, 2) \cup (6, +\infty)$
f is decreasing on $(2, 6)$.

(c) [**2 points**] Find the possible points of inflection (PPI) of f. Evaluate f at each of the PPI, and list which of the PPI are actual inflection points. Where is f concave up, and where is it concave down?

Solution. The PPI of f are those x for which either $f''(x) = 0$ or $f''(x)$ is undefined. Note that $f''(x)$ is never zero, and $f''(x)$ is undefined when $x = 0$ and when $x = 6$. We determine the concave up/down behavior of f, and we decide which of these are actual inflection points, by testing the sign of f'':



Where we have evaluated f'' at the test points:

$$f''(-1) = -\frac{8}{(-1)^{\frac{5}{3}}(6-(-1))^{\frac{4}{3}}} = \frac{8}{7^{\frac{4}{3}}} > 0$$
$$f''(1) = -\frac{8}{(1)^{\frac{5}{3}}(6-(1))^{\frac{4}{3}}} = -\frac{8}{5^{\frac{4}{3}}} < 0$$
$$f''(10) = -\frac{8}{(10)^{\frac{5}{3}}(6-(10))^{\frac{4}{3}}} = -\frac{8}{10^{\frac{5}{3}}(-4)^{\frac{4}{3}}} < 0.$$

We thus know that f is concave up on $(-\infty, -1)$ and concave down on $(1, +\infty)$, and that $x = 0$ is an inflection point of f, whereas $x = 6$ is not. We also already know that $f(0) = 0 = f(6)$, and so in summary:

(0,0) is a point of inflection
(6,0) is not a point of inflection

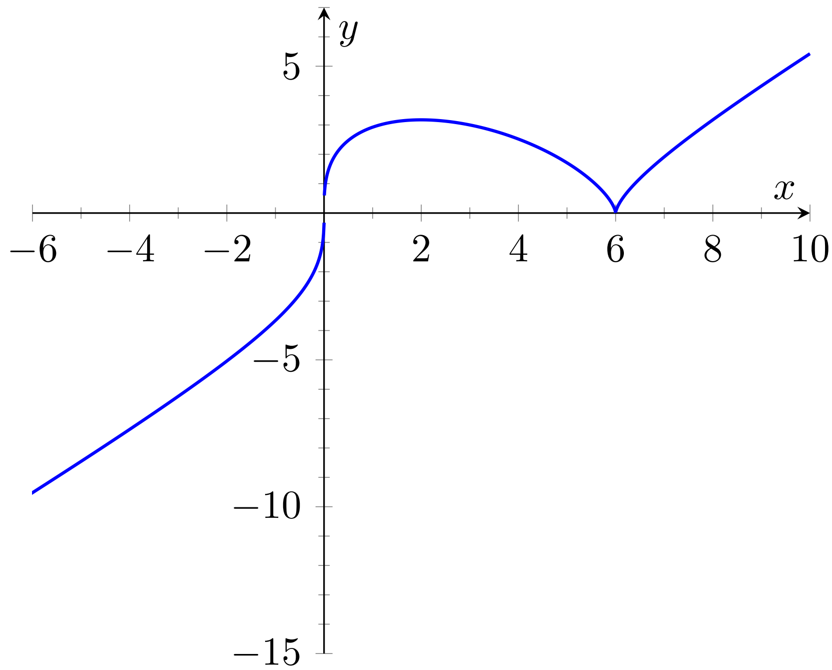
and

f is concave up on $(-\infty, 0)$
f is concave down on $(0, +\infty)$.

(d) [**1 point**] Find the x and y intercepts of f.

Solution. The y-intercept is given by $f(0) = 0$—that is, by the point $(0,0)$. The x-intercepts are the solutions of $f(x) = 0$; that is, of $0 = x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}$. This has only the solutions $x = 0$ and $x = 6$, so the x-intercepts are $(0,0)$ and $(6,0)$.

(e) [3 points] Without the aid of calculators or computational tools, sketch the graph $y = f(x)$.



Problem 3. Let $f(x) = \frac{2x^2+1}{x^2-2x}$.

(a) **[1 point]** Find the first derivative $f'(x)$ and the second derivative $f''(x)$.

Solution.

$$\begin{aligned}
f'(x) &= \frac{(x^2-2x)\frac{\mathrm{d}}{\mathrm{d}x}[2x^2+1] - (2x^2+1)\frac{\mathrm{d}}{\mathrm{d}x}[x^2-2x]}{(x^2-2x)^2} \\
&= \frac{(x^2-2x)4x^2 - (2x^2+1)(2x-2)}{x^2(x-2)^2} \\
&= \frac{4x^3 - 8x^2 - 4x^3 + 4x^2 - 2x + 2}{x^2(x-2)^2} \\
&= \frac{-4x^2 - 2x + 2}{x^2(x-2)^2} \\
&= -2\frac{(2x-1)(x+1)}{x^2(x-2)^2} \\
f''(x) &= -2\frac{\mathrm{d}}{\mathrm{d}x}\left[2x^2+x-1\right]x^{-2}(x-2)^{-2} - 2(2x^2+x-1)\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-2}\right](x-2)^{-2} - \\
&\qquad - 2(2x^2+x-1)x^{-2}\frac{\mathrm{d}}{\mathrm{d}x}\left[(x-2)^{-2}\right] \\
&= -2(4x+1)x^{-2}(x-2)^{-2} - 2(2x^2+x-1)(-2x^{-3})(x-2)^{-2} - \\
&\qquad - 2(2x^2+x-1)x^{-2}(-2(x-2)^{-3}\frac{\mathrm{d}}{\mathrm{d}x}[x-2]) \\
&= \frac{-8x-2}{x^2(x-2)^2} + \frac{8x^2+4x-4}{x^3(x-2)^2} + \frac{8x^2+4x-4}{x^2(x-2)^3} \\
&= \frac{-2x(x-2)(4x+1) + (8x^2+4x-4)(x-2) + (8x^2+4x-4)x}{x^3(x-2)^3} \\
&= 2\frac{-x(x-2)(4x+1) + (8x^2+4x-4)(x-1)}{x^3(x-2)^3} \\
&= 2\frac{-4x^3 + 7x^2 + 2x + 8x^3 + 4x^2 - 4x - 8x^2 - 4x + 4}{x^3(x-2)^3} \\
&= 2\frac{4x^3 + 3x^2 - 6x + 4}{x^3(x-2)^3}.
\end{aligned}$$

(b) [3 points] Find and classify the critical points of f. Where is the line tangent to f horizontal, where is it vertical, and where does it fail to exist? Evaluate $f(x)$ at each of the critical points. Where is f increasing, and where is it decreasing?

Solution. The critical points of f are those x for which $f'(x) = 0$, or for which $f'(x)$ is undefined. Note that $f'(-1) = 0$ and $f'(0.5) = 0$ so $x = -1$ and $x = 0.5$ are both critical point where f has a horizontal tangent line. Also, $f'(0)$ and $f'(2)$ are both undefined, but $f(0)$ and $f(2)$ also do not exist; thus, $x = 0$ and $x = 2$ are both critical points where the tangent line does not exist. Now, let's determine the increasing/decreasing behavior of f, and the local extrema by using the first derivative test:

x	-2	-1	-0.5	0	0.25	0.5	1	2	5
	Test Point	Critical Point	Test Point	Critical Point	Test Point	Critical Point	Test Point	Critical Point	Test Point
$f'(x)$	$- - - - - - -$		$+ + + + + + +$		$+ + + + + + +$		$- - - - - - -$		$- - - - - - -$

Where we have evaluated f' at the test points:

$$
\begin{aligned}
f'(-2) &= \frac{-2\,[2(-2)-1]\,[(-2)+1]}{(-2)^2\,[(-2)-2]^2} < 0 \\
f'(-0.5) &= \frac{-2\,[2(-0.5)-1]\,[(-0.5)+1]}{(-0.5)^2\,[(-0.5)-2]^2} > 0 \\
f'(0.25) &= \frac{-2\,[2(0.25)-1]\,[(0.25)+1]}{(0.25)^2\,[(0.25)-2]^2} > 0 \\
f'(1) &= \frac{-2\,[2(1)-1]\,[(1)+1]}{(1)^2\,[(1)-2]^2} < 0 \\
f'(5) &= \frac{-2\,[2(5)-1]\,[(5)+1]}{(5)^2\,[(5)-2]^2} < 0.
\end{aligned}
$$

We thus know that f is increasing on $(-1, 0.5)$, and is decreasing on $(-\infty, -1) \cup (0.5, +\infty)$. We also know that $x = 0$ and $x = 2$ are not extrema, that $x = 0.5$ gives a local maximum, and $x = -1$ gives a local minimum of f. We evaluate f at each of the critical points:

$$f(-1) = 1; \qquad f(0.5) = -2; \qquad f(0) \text{ and } f(2) \text{ do not exist.}$$

Thus, in summary:

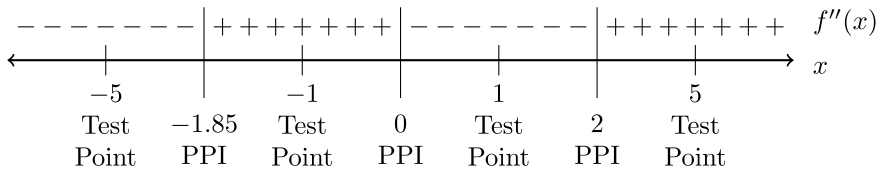
(-1,1)	is a local minimum, and	has a horizontal tangent line
at $x = 0$	neither the function value	nor the tangent line exist
(0.5,-2)	is a local maximum, and	has a horizontal tangent line
at $x = 2$	neither the function value	nor the tangent line exist

and

f is increasing	on	$(-1, 0.5)$
f is decreasing	on	$(-\infty, -1) \cup (0.5, +\infty)$.

(c) **[2 points]** Find the possible points of inflection of f. Evaluate f at each of the possible points of inflection, and list which of them are actually points of inflection. Where is f concave up, and where is it concave down?

Solution. The PPI of f are those x for which either $f''(x) = 0$ or $f''(x)$ is undefined. Note that $f''(x)$ is zero only when $4x^3 + 3x^2 - 6x + 4 = 0$, and $f''(x)$ is undefined when $x = 0$ and when $x = 2$. The only real zero of the polynomial $4x^3 + 3x^2 - 6x + 4$ is approximately -1.85 (we got this value using WolframAlpha; we could have used Newton's method, and that would have been completely awesome of us! But we already had a lot to do, so...). We determine the concave up/down behavior of f, and we decide which of these are actual inflection points, by testing the sign of f'':



Where we have evaluated f'' at the test points:

$$\begin{aligned}
f''(-5) &= 2\frac{4(-5)^3 + 3(-5)^2 - 6(-5) + 4}{(-5)^3\left[(-5) - 2\right]^3} < 0 \\
f''(-1) &= 2\frac{4(-1)^3 + 3(-1)^2 - 6(-1) + 4}{(-1)^3\left[(-1) - 2\right]^3} > 0 \\
f''(1) &= 2\frac{4(1)^3 + 3(1)^2 - 6(1) + 4}{(1)^3\left[(1) - 2\right]^3} < 0 \\
f''(5) &= 2\frac{4(5)^3 + 3(5)^2 - 6(5) + 4}{(5)^3\left[(5) - 2\right]^3} > 0.
\end{aligned}$$

We thus know that f is concave up on $(-1.85, 0) \cup (2, +\infty)$ and concave down on $(-\infty, -1.85) \cup (0, 2)$, and that $x = -1.85$, $x = 0$, and $x = 2$ are all inflection points of f. We also already know that $f(0)$ and $f(2)$ do not exist, and we can calculate that $f(-1.85) \approx 1.102$, so in summary:

$$\begin{aligned}
(-1.85, 1.102) &\quad \text{is a point of inflection} \\
x = 0 &\quad \text{is a point of inflection} \\
x = 2 &\quad \text{is a point of inflection,}
\end{aligned}$$

and

$$\begin{aligned}
f \text{ is concave up} &\quad \text{on} \quad (-1.85, 0) \cup (2, +\infty) \\
f \text{ is concave down} &\quad \text{on} \quad (-\infty, -1.85) \cup (0, 2).
\end{aligned}$$

(d) **[2 points]** Locate all vertical, horizontal, and/or slant asymptotes of f.

Solution. Note that

$$\lim_{x\to\pm\infty} \frac{2x^2+1}{x^2-2x} = \lim_{x\to\pm\infty} \frac{2+\frac{1}{x^2}}{1-\frac{2}{x}} = 2,$$

so that $y = 2$ is a horizontal asymptote of f.

Note also that

$$\lim_{x\to 0^+} \frac{2x^2+1}{x(x-2)} = -\infty, \quad \text{and} \quad \lim_{x\to 0^-} \frac{2x^2+1}{x(x-2)} = +\infty,$$

and

$$\lim_{x\to 2^+} \frac{2x^2+1}{x(x-2)} = +\infty, \quad \text{and} \quad \lim_{x\to 2^-} \frac{2x^2+1}{x(x-2)} = -\infty,$$

so that $x = 0$ and $x = 2$ are both vertical asymptotes of f.

Because f is a rational function whose denominator is a polynomial of the same degree as the polynomial in the numerator, we do not expect that f will have any slant asymptotes.

(e) **[2 points]** **Without the aid of calculators or computational tools**, sketch the graph $y = f(x)$.