Lecture 9: June 19.

Announcements/ Assignments.

- Webwork 8 due Friday
- Webwork 9 due Monday
- Homework 4 due Monday

Today.

- 8.1: Using Basic Integration Formulas.
- 8.2: Integration by Parts.

8.1 : Basic Integration Formulas.

Page 457, table 8.1

22 integration formulas, some we already know...

$$\int k\,dx = kx + C \qquad \int \cos x\,dx = \sin(x) + C$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad \int \sin x\,dx = -\cos(x) + C$$

$$\int \frac{dx}{x} = \ln|x| + C \qquad \int e^x\,dx = e^x + C$$

... and some that may be less familiar:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \arccos\left(\frac{x}{a}\right) + C, \quad \text{for } x > a > 0$$

etc.

Put these formulas in your notebooks!

(And on a reference sheet for the final exam)

$\boxed{\text{Example } 1, \text{ p. } 456}$ Evaluate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx =: I$

Let $u := x^2 - 3x + 1$, $du = 2x - 3\,dx$.

$u(3) = 3^2 - 3(3) + 1 = 1$, $u(5) = 5^2 - 3(5) + 1 = 11$

Then $I = \displaystyle\int_1^{11} \frac{du}{\sqrt{u}} = \int_1^{11} u^{-1/2}\,du$

$\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$

$$= \frac{u^{1/2}}{1/2}\Bigg|_1^{11}$$

$$= 2\sqrt{u}\,\Big|_1^{11}$$

$$= 2\left(\sqrt{11} - \sqrt{1}\right)$$

$$\boxed{= 2\sqrt{11} - 2} \approx 4.63$$

Example 2, p. 457 Evaluate $\int \frac{dx}{\sqrt{8x-x^2}} =: I$

(Hint: complete the square.)

Recall: Completing the square.

$$(a+b)^2 = a^2 + 2ab + b^2$$

① Solve for b

$2b = 8 \Rightarrow b = 4$

$$8x - x^2 = -(x^2 - 8x)$$

$$= -(x^2 - 8x - 16 + 16)$$

② Add b^2 & Subtract b^2.

$$= -\underbrace{\left((x-4)^2 - 16\right)}$$

$$(x-4)^2 - 16 = x^2 - 8x + 16 - 16$$

$$= -\left((x-4)^2 - 16\right) = 16 - (x-4)^2$$

So $I = \int \frac{dx}{\sqrt{16-(x-4)^2}}$

Know: $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$

Let $u := x-4$, $du = dx$

$$= \int \frac{du}{\sqrt{16-u^2}} = \arcsin\left(\frac{u}{4}\right) + C = \boxed{\arcsin\left(\frac{x-4}{4}\right) + C}$$

Example 3, p. 458

Evaluate $\int (\cos x \sin 2x + \sin x \cos 2x)\, dx$.

Hint: use sine addition formula:

$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b).$$

$$I = \int \sin(x+2x)\, dx = \int \sin(3x)\, dx$$

$u = 3x$

$du = 3\,dx \Leftrightarrow dx = \frac{1}{3} du$

$$= \int \sin(u) \tfrac{1}{3}\, du$$

$$= \frac{1}{3}(-\cos u) + C$$

$$= \boxed{-\frac{1}{3}\cos(3x) + C}$$

$\boxed{\text{Example } 4, \text{ p. } 458}$ Evaluate $\displaystyle\int_0^{\pi/4} \frac{dx}{1-\sin x} \underbrace{\left(\frac{1+\sin x}{1+\sin x}\right)}_{=1}$

$$I = \int_0^{\pi/4} \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\, dx$$

$$= \int_0^{\pi/4} \frac{1+\sin x}{1-\sin^2 x}\, dx$$

$\cos^2 x + \sin^2 x = 1 \quad \forall x$

$1 - \sin^2 x = \cos^2 x$

$$= \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\, dx$$

$$= \int_0^{\pi/4} \sec^2 x + \tan x \sec x \, dx$$

$$= \Big[\tan x + \sec x\Big]_0^{\pi/4}$$

$$= \tan(\pi/4) + \sec(\pi/4) - \big(\tan(0) + \sec(0)\big)$$

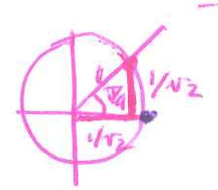
$$= \frac{\sin(\pi/4)}{\cos(\pi/4)} + \frac{1}{\cos(\pi/4)} - \left(\frac{\sin(0)}{\cos(0)} + \frac{1}{\cos(0)}\right)$$

$$= 1 + \sqrt{2} - 0 - 1$$

$\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$

$= \sqrt{2}/2$

$$\boxed{= \sqrt{2}}$$



$\boxed{\text{Example } 5, \text{p.}458}$ Evaluate $\int \frac{3x^2-7x}{3x+2} =: I$

Recall: Polynomial division.

$$\begin{array}{r|l} & x - 3 \\ \hline 3x+2 & 3x^2 - 7x \\ & -(3x^2+2x) \\ \hline & -9x + 0 \\ & -(-9x-6) \\ \hline & 6 \end{array}$$

Therefore, $\frac{3x^2-7x}{3x+2} = x-3 + \frac{6}{3x+2}$

$$I = \int x-3 + \frac{6}{3x+2}\, dx$$

$$= \underbrace{\int x-3\, dx} + 6\int \frac{1}{3x+2}\, dx$$

Let $u := 3x+2$, $du = 3dx$
$\Rightarrow 6\,dx = 2du$

$$= \frac{1}{2}x^2 - 3x + c_1 + \int \frac{2\,du}{u}$$

$$= \frac{1}{2}x^2 - 3x + c_1 + 2\ln|u| + c_2$$

$$= \boxed{\frac{1}{2}x^2 - 3x + 2\ln|3x+2| + C}$$

$C := c_1 + c_2$

Example 6, p. 459 Evaluate $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx =: I$

$$I = \underbrace{\int \frac{3x}{\sqrt{1-x^2}}\,dx} + \int \frac{2}{\sqrt{1-x^2}}\,dx$$

Recall: $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$

Let $u := 1-x^2$

$du = -2x\,dx$

$\Rightarrow 3x\,dx = -\frac{3}{2}\,du$

$$= \int -\frac{3}{2}\frac{1}{\sqrt{u}}\,du + 2\arcsin(x) + C$$

$$= -\frac{3}{2}\int u^{-1/2}\,du + 2\arcsin(x) + C$$

$\frac{1}{1/2} = 2$

$$= -\frac{3}{2}(2)\sqrt{u} + 2\arcsin(x) + C$$

$$= \boxed{-3\sqrt{1-x^2} + 2\arcsin(x) + C}$$

$\boxed{\begin{array}{c}\text{Example} \\ 7, \text{p.} 459\end{array}}$ Evaluate $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3} =: I$

Let $u := 1 + \sqrt{x}$. $du = \dfrac{1}{2\sqrt{x}}\,dx$

$\boxed{\sqrt{x} = u - 1}$ $\rightarrow dx = 2\sqrt{x}\,du$

$dx = 2(u-1)\,du$

$$I = \int \frac{2(u-1)\,du}{u^3}$$

$$= \int \frac{2u}{u^3}\,du - \int \frac{2}{u^3}\,du$$

$$= \int \frac{2}{u^2}\,du - \int \frac{2}{u^3}\,du$$

$$= \int 2u^{-2}\,du - \int 2u^{-3}\,du$$

$$= 2(-1)u^{-1} - 2\left(-\frac{1}{2}\right)u^{-2} + C$$

$$= -\frac{2}{u} + \frac{1}{u^2} + C$$

$$= \frac{1-2u}{u^2} + C$$

$$= \frac{1-2(1+\sqrt{x})}{(1+\sqrt{x})^2} + C = \boxed{\frac{-2\sqrt{x}-1}{(1+\sqrt{x})^2} + C}$$

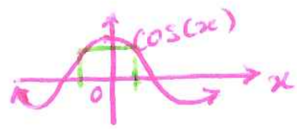
Example 8, p. 460

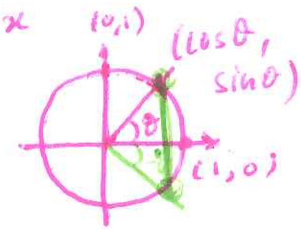
Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx$.

Note: ① $[-\pi/2, \pi/2]$ is symmetric abt. $x = 0$.

② $x^3 \cos x$ is even? odd? function

Let $f(x) := x^3 \cos(x)$.





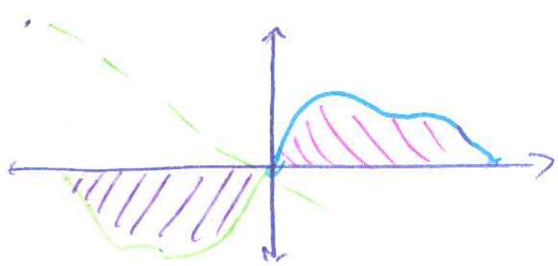
Check: $f(-x) = (-x)^3 \cos(-x)$

$= -x^3 \cos(x)$

$= -f(x)$.

when $f(-x) = -f(x)$, the function is called odd.

For odd fns. integrated over an interval symmetric about 0 ... the integral is 0.



$$\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx = 0.$$

8.2: Integration by Parts.

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)\,g(x)\,dx,$$

where
- f is easily differentiable (repeatedly)
- g is easily integrable (repeatedly)

For example,

$$\int x\cos x\,dx$$

- $f(x) = x$
- $g(x) = \cos(x)$

$$\int x^2 e^x\,dx$$

- $f(x) = x^2$
- $g(x) = e^x$

Another example:

$$\int \ln(x)\,dx.$$

- $f(x) = \ln(x)$
- $g(x) = 1$

We will actually integrate these later...

8.2: Integration by Parts, ct'd.

Recall the Product Rule for differentiation:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x),$$

Then for indefinite integrals,

$$\int \frac{d}{dx}\left[f(x)g(x)\right]dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

By the Fundamental theorem of Calculus,

$$\int \frac{d}{dx}\left[f(x)g(x)\right]dx = f(x)g(x)$$

Then

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx,$$

and if f is easily differentiable, then we rearrange:

$$\int \underbrace{f(x)}_{u}\,\underbrace{g'(x)\,dx}_{dv} = \underbrace{f(x)}_{u}\,\underbrace{g(x)}_{v} - \int \underbrace{\underbrace{f'(x)}\,\underbrace{g(x)}_{v}\,\underbrace{dx}}_{du}.$$

Let $u := f(x)$ $\quad v := g(x)$

$du = f'(x)\,dx$ $\quad dv := g'(x)\,dx$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

MEMORIZE THIS

Example 1, p. 462 Evaluate $\boxed{\int x \cos x \, dx.}$

Sol. $\int u \, dv = uv - \int v \, du.$

$u := x$ $\quad v = \sin x$

$du = dx$ $\quad dv := \cos x \, dx$

* HOW TO CHOOSE u AND dv ?

$\int v \, du$ should be easier than $\int u \, dv$.

$$I = \int x \cos x \, dx$$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x - (-\cos x) + C$$

$$= \boxed{x \sin x + \cos x + C}$$

Example 2, p.463

Evaluate $\int \ln x \, dx$.

$$\int u \, dv = uv - \int v \, du$$

$u := \ln x$ $\qquad$ $v = x$

$du = \frac{1}{x} dx$ $\qquad$ $dv := 1 dx$

$$\int \ln(x) \, dx = x \ln x - \int \frac{x}{x} \, dx$$

$$= x \ln x - \int 1 \cdot dx$$

$$= x \ln x - x + c$$

$$\boxed{= x(\ln x - 1) + c}$$

Example. 3, p.463

Evaluate $\int x^2 e^x \, dx$.

$$\int u \, dv = uv - \int v \, du.$$

$u := x^2$ $v = e^x$

$du = 2x \, dx$ $dv := e^x \, dx$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

$$= x^2 e^x - 2 \int x e^x \, dx$$

$u := x$ $v = e^x$

$du = dx$ $dv := e^x dx$

$$\int x^2 e^x \, dx = x^2 e^x - 2 \left[x e^x - \int e^x \, dx \right]$$

$$= x^2 e^x - 2x e^x + 2 \int e^x \, dx$$

$$= x^2 e^x - 2x e^x + 2 e^x + C$$

$$\boxed{= e^x (x^2 - 2x + 2) + C}$$

Example 9, p. 464 Evaluate $\int e^x \cos x \, dx$.

$$\int u \, dv = uv - \int v \, du$$

$u := \cos x \qquad v = e^x$

$du = -\sin x \, dx \qquad dv := e^x \, dx$

$$\int e^x \cos x \, dx = e^x \cos x - \int e^x (-\sin x \, dx)$$

$$\int e^x \cos x \, dx = e^x \cos x + \underline{\int e^x \sin x \, dx}$$

$u := \sin x \qquad v = e^x$

$du = \cos x \, dx \qquad dv := e^x \, dx$

$$\int e^x \cos x \, dx = e^x \cos x + \left[e^x \sin x - \int e^x \cos x \, dx \right]$$

$$\underbrace{\int e^x \cos x \, dx}_{=: I} = e^x \cos x + e^x \sin x - \underbrace{\int e^x \cos x \, dx}_{I}$$

$$2I = e^x \cos x + e^x \sin x$$

$$I = \boxed{\tfrac{1}{2} e^x (\cos x + \sin x) + C}$$

Example. 5, p. 464

Obtain a formula that expresses

$$\int (\cos x)^m \, dx$$

in terms of an integral of a lower power of $(\cos x)$.

$$\int (\cos x)^m \, dx = \int \cos x \cdot (\cos x)^{m-1} \, dx$$

$$\int u \, dv = uv - \int v \, du$$

$$u := (\cos x)^{m-1} \qquad v = \sin x$$

$$dv := \cos x \, dx$$

$$du = (m-1)(\cos x)^{m-2}(-\sin x) \, dx$$

$$= (1-m) \sin x (\cos x)^{m-2} \, dx$$

$$\int (\cos x)^m \, dx = \sin x (\cos x)^{m-1} - \int (1-m)\underbrace{(\sin x)^2}_{} (\cos x)^{m-2} \, dx$$

$\cos^2 x + \sin^2 x = 1$

$\Rightarrow \sin^2 x = 1 - \cos^2 x$

$$\int (\cos x)^m \, dx = \sin x (\cos x)^{m-1} + (m-1) \int (1 - \cos^2 x)(\cos x)^{m-2} \, dx$$

$$\int (\cos x)^m \, dx = \sin x (\cos x)^{m-1} + (m-1) \int (\cos x)^{m-2} - (\cos x)^m \, dx$$

$$\underbrace{\int (\cos x)^m \, dx}_{I :=} = \sin x (\cos x)^{m-1} + (m-1) \int (\cos x)^{m-2} \, dx + (1-m) \underbrace{\int (\cos x)^m \, dx}_{I}$$

$$\boxed{I = \frac{1}{m} \sin x (\cos x)^{m-1} + \frac{(m-1)}{m} \int (\cos x)^{m-2} \, dx}$$

Can also evaluate definite integrals by parts.

$\boxed{\begin{array}{l}\text{Example.}\\ 6,\ \text{p. } 465\end{array}}$ Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x=0$ to $x=4$.

$$\int_0^4 xe^{-x}\,dx.$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$u := x \qquad v = -e^{-x}$

$du = dx \qquad dv := e^{-x}\,dx$

$$\int_0^4 xe^{-x}\,dx = -xe^{-x}\Big|_0^4 - \int_0^4 (-e^{-x})\,dx$$

$$= -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}\,dx$$

$$= \Big[-xe^{-x} - e^{-x}\Big]_0^4$$

$$= \underbrace{-4e^{-4} - e^{-4} - \left[-0\cdot e^{-0} - e^{-0}\right]}$$

$$= e^{-4}(-4-1) + 1$$

$$\boxed{= 1 - 5e^{-4}}$$

$$\approx 0.91$$

Example. 8, p. 466

Evaluate $\overbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}^{=:\, I}$, where

$$f(x) = \begin{cases} 1, & x \in [-\pi, 0) \\ x^3, & x \in [0, \pi], \end{cases} \quad \text{and } n \in \mathbb{N}.$$

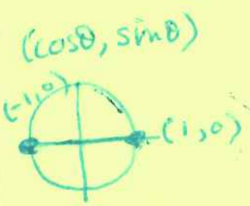
$$I = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos(mx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos(mx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \cos(mx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x^3\cos(mx)\,dx$$

$\int u\,dv = uv - \int v\,du$

$u := x^3 \qquad v = \frac{1}{n}\sin(nx)$

$du = 3x^2\,dx \qquad dv := \cos(mx)\,dx$



$$= \underbrace{\frac{1}{\pi}\left[\frac{1}{m}\sin(mx)\right]_{-\pi}^{0}} + \frac{1}{\pi}\left[\frac{x^3}{m}\sin(mx)\Big|_0^{\pi} - \int_0^{\pi}\frac{3}{m}x^2\sin(mx)\,dx\right]$$

$$= \frac{1}{m\pi}\Big[\underbrace{\sin(m\cdot 0)}_{0} - \underbrace{\sin(-m\pi)}_{0}\Big] + \frac{x^3}{m\pi}\Big[\underbrace{\sin(m\pi)}_{0} - \underbrace{\sin(m\cdot 0)}_{0}\Big] - \frac{3}{m\pi}\int_0^{\pi} x^2\sin(mx)\,dx$$

$$= -\frac{3}{m\pi}\int_0^{\pi} x^2\sin(mx)\,dx$$

$u := x^2 \qquad v = -\frac{1}{m}\cos(mx)$

$du = 2x\,dx \qquad dv := \sin(mx)\,dx$

$$= -\frac{3}{m\pi}\left[-\frac{x^2}{m}\cos(mx)\Big|_0^{\pi} - \int_0^{\pi}\left(-\frac{2x}{m}\cos(mx)\right)dx\right]$$

$$= \frac{3x^2}{m^2\pi}\left[\cos(m\pi) - \cos(0)\right] - \frac{6x}{m^2\pi}\int_0^{\pi} x\cos(mx)\,dx$$

$u := x \qquad v = \frac{1}{m}\sin(mx)$

$du = dx \qquad dv := \cos(mx)\,dx$

$$= \frac{3x^2}{m^2\pi}\left[(-1)^m - 1\right] - \frac{6x}{m^2\pi}\left[\frac{x}{m}\sin(mx)\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{m}\sin mx\,dx\right]$$

$$= \frac{3x^2}{n^2\pi}\left[(-1)^n - 1\right] - \frac{6x^2}{n^3\pi}\left[\underbrace{\sin(\pi x)}_{0} - \underbrace{\sin(0)}_{0}\right] + \frac{6x}{n^3\pi}\left(-\frac{1}{n}\cos(nx)\right)\Big|_0^{\pi}$$

$$0$$

$$= \frac{3x^2}{n^2\pi}\left[(-1)^n - 1\right] - \frac{6x}{n^4\pi}\left[\cos(n\pi) - \cos(0)\right]$$

$$= \left[(-1)^n - 1\right]\left(\frac{3x^2}{n^2\pi} - \frac{6x}{n^4\pi}\right)$$

$$= \left[(-1)^n - 1\right]\left(\frac{3x^2n^2 - 6x}{n^4\pi}\right).$$

when n is <u>even</u>, $(-1)^n - 1 = 1 - 1 = 0$.

when n is odd, $(-1)^n - 1 = -1 - 1 = -2$

$$I = \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{-2(3x^2n^2 - 6x)}{n^4\pi}, & \text{when } n \text{ is odd.} \end{cases}$$