Calculus II
E1 Term, Sections E101 and E196
Instructor: E.M. Kiley
Due May 23, 2016

Week 1: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it. An example of acceptable homework solutions is posted on myWPI under "Course Materials".

Reading

Please read Sections 4.8 and 5.1–5.2 in time for Tuesday's lecture, and Sections 5.3 and 5.4 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 4, "Questions to Guide Your Review", p. 291, Problems 21, 23, and 24
- Chapter 5, "Questions to Guide Your Review", p. 529, Problems 1–3 and 5–6

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 4.8, Problems 1; 7; 11–23 odd; 91–95 odd
- Section 5.1, Problems 1–7 odd; 15; 17
- Section 5.2, Problems 1–29 odd; 33–45 odd
- Section 5.3, Problems 1–13 odd; 37–47 odd; 51–61 odd
- Section 5.4, Problems 1–33 odd; 39–55 odd; 65–69

[1]See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

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Week 1: Homework Problems

Due date: Monday, May 23, 2015, 11:59 p.m. EDT. Please upload a `.pdf` version to myWPI (`my.wpi.edu`).

Rules for Calculus Assignments:

I) Each student must compose his or her assignments independently. However, brainstorming may be done in groups.

II) Please typeset your solutions using LaTeX, or handwrite them neatly and legibly using correct English.

III) Show your work. Explain your answers using **full English sentences**.

IV) **No late assignments will be accepted for credit.**

Problem 1. In a vacuum, all bodies fall with the same constant acceleration. To demonstrate this, the Apollo 15 astronaut David Scott dropped a hammer and a feather on the moon from about 4 feet above the ground. The television footage of the event, which you can find at `https://commons.wikimedia.org/wiki/File:Apollo_15_feather_and_hammer_drop.ogg`, shows the hammer and the feather falling more slowly than they do on Earth, where, in a vacuum, they would have taken only half a second to fall 4 ft.

(a) [**8 points**] Solve the following initial value problem for s as a function of t.

$$\begin{cases} \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -5.2 \text{ ft/sec}^2 \\ \left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=0} = 0 \text{ ft/sec}, \quad s(0) = 4 \text{ ft.} \end{cases}$$

For the feather-and-hammer problem, $s(t)$ represents the position of the hammer and feather above the ground at time t, and $t = 0$ represents the time when Colonel Scott released the hammer and feather. Hint: you will solve the given initial value problem for $\frac{\mathrm{d}s}{\mathrm{d}t}$, and this will give you *another* initial value problem to solve for $s(t)$.

Solution. The initial value problem to solve for $\frac{\mathrm{d}s}{\mathrm{d}t}$ is:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{t}}\left[\frac{\mathrm{d}s}{\mathrm{d}t}\right] = -5.2 \text{ ft/sec}^2 \\ \left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=0} = 0 \text{ ft/sec,} \end{cases}$$

which comes directly from the given second-order initial value problem.
We solve this problem for $\frac{\mathrm{d}s}{\mathrm{d}t}$. First, the general family of antiderivatives of -5.2 can be written as $-5.2t + C$, where C is an arbitrary constant. Then the general solution to the initial value problem is

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -5.2t + C.$$

Now, we attempt to solve for C. On the one hand, from the general solution we have

$$\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=0} = -5.2(0) + C = C,$$

and on the other hand, the condition given in the initial value problem was $\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=0} = 0$. These two facts, taken together, imply that $C = 0$, and so we have

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -5.2t.$$

Now, we may write a second initial value problem that we will solve for $s(t)$:

$$\begin{cases} \frac{\mathrm{d}s}{\mathrm{d}t} = -5.2t, \\ s(0) = 4 \text{ ft.} \end{cases}$$

To solve this initial value problem, we first note that the general family of antiderivatives of $-5.2t$ can be written as $\frac{-5.2}{2}t^2 + D = -2.6t^2 + D$, where D is an arbitrary constant. Then the general solution to the initial value problem is

$$s(t) = -2.6t^2 + D.$$

Now, we attempt to solve for D. On the one hand, from the general solution we have

$$s(0) = -2.6(0)^2 + D = D,$$

and on the other hand, the condition given in the initial value problem was $s(0) = 4$. Taken together, these two facts imply that $D = 4$, and so the solution of $s(t)$ is

$$s(t) = -2.6t^2 + 4.$$

(b) **[2 points]** Call the time when the hammer and feather hit the ground t_{hit}. Find a value for t_{hit}, which satisfies $s(t_{\text{hit}}) = 0$.

Solution. If $s(t_{\text{hit}}) = 0$, then $-2.6(t_{\text{hit}})^2 + 4 = 0$, and so

$$t_{\text{hit}}^2 = \frac{4}{2.6} = \frac{2}{1.3}, \text{ which is true if and only if } t_{\text{hit}} = \pm\sqrt{\frac{2}{1.3}} \approx \pm 1.2403.$$

This implies that there are two solutions to this problem; however, since t represents time since the hammer and feather were released, only positive values of t make sense for us. This means that at time $t = 1.2403$ seconds, the hammer and feather both hit the ground (contrast to the time that it would take on Earth: about half a second).

Problem 2. [6 points] Suppose that the differentiable functions $y = F(x)$ and $y = G(x)$ both solve the initial value problem

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = f(x), \\ y(x_0) = y_0 \end{cases}$$

for $x \in I$, an interval of real numbers.

(a) [**4 pt**] What is an initial value problem that the function $H(x) := F(x) - G(x)$ solves? [Hint: Compute $\frac{\mathrm{d}H}{\mathrm{d}x}$, and evaluate $H(x_0)$. Write in initial value problem form (using y as the "unknown" function).]

Solution. Computing $\frac{\mathrm{d}H}{\mathrm{d}x}$, we obtain

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[F(x) - G(x)] = \frac{\mathrm{d}F}{\mathrm{d}x} - \frac{\mathrm{d}G}{\mathrm{d}x} = f(x) - f(x) = 0,$$

and when we compute $H(x_0)$, we obtain

$$H(x_0) = [F - G](x_0) = F(x_0) - G(x_0) = y_0 - y_0 = 0.$$

So an initial value problem solved by $H(x)$ is

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = 0, \\ y(x_0) = 0. \end{cases}$$

(b) [**2 pt**] Solve the initial value problem you found previously, to find out what $H(x)$ has to be. What does this mean about uniqueness of solutions to initial value problems?

Solution. The general family of antiderivatives of 0 is represented by $0 + C = C$, where C is an arbitrary constant. This implies that the general solution is $y(x) = C$. To solve for the constant, observe that on the one hand, $y(x_0) = C$, and on the other hand, $y(x_0) = 0$. Taken together, these two facts imply that $C = 0$, and so $y(x) \equiv 0$ is the solution of this initial value problem.

If the only solution to the initial value problem that $H(x)$ solves is 0, then $H(x) \equiv 0$, and so $F(x) - G(x) = 0$. This implies that if F and G solve the same first-order initial value problem, then it must be the case that $F(x) = G(x)$, and so solutions to first-order initial value problems, if they exist, are unique.

Problem 3. [5 pt] What is the definite integral of $f(x) = x^3$ over the interval $[0, 1]$?

Solution. The definite integral is

$$\int_0^1 x^3 \, \mathrm{d}x = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4 = \frac{1}{4}.$$

Problem 4. Using rectangles whose height is given by the value of the function at the *midpoint* of each rectangle, estimate the area under the graph of $f(x) = x^3$ between $x = 0$ and $x = 1$, using.... What is the error in each case? (That is, what is the absolute value of the definite integral minus the area estimate?)

(a) [**4 pt**] Two rectangles

Solution. Subdividing the domain, we have the partition $S = \left\{0, \frac{1}{2}, 1\right\}$. The midpoint of the first interval is

$$m := \frac{0 + \frac{1}{2}}{2} = \frac{1}{4},$$

so the height of the first rectangle is

$$f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^3 = \frac{1}{64}.$$

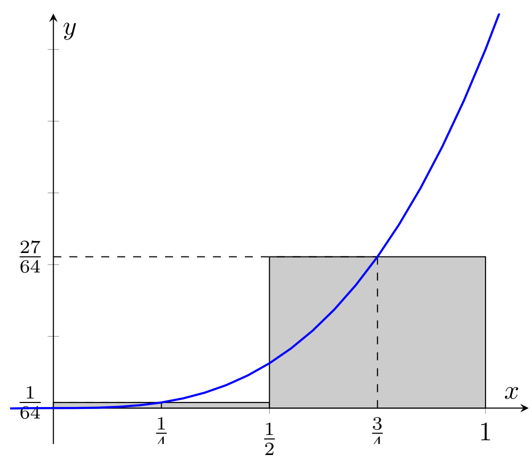
The midpoint of the second interval is

$$m := \frac{\frac{1}{2} + 1}{2} = \frac{3}{4},$$

so the height of the second rectangle is

$$f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^3 = \frac{27}{64}.$$

Draw the two rectangles on the graph of $y = x^3$ as shown:



The sum of the area of the rectangles is:

$$A = \frac{1}{2} \cdot \frac{1}{64} + \frac{1}{2} \cdot \frac{27}{64} = \frac{1 + 27}{2 \cdot 64} = \frac{28}{2 \cdot 64} = \frac{14}{64} = \frac{7}{32} \approx 0.2188.$$

Error is $\left|\frac{1}{4} - 0.2188\right| \approx 0.0312$.

(b) [**5 pt**] Four rectangles

Solution. Subdividing the domain, we have the partition $S = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$. The midpoint of the first interval is

$$m := \frac{0 + \frac{1}{4}}{2} = \frac{1}{8},$$

so the height of the first rectangle is

$$f\left(\frac{1}{8}\right) = \left(\frac{1}{8}\right)^3 = \frac{1}{512}.$$

The midpoint of the second interval is

$$m := \frac{\frac{1}{4} + \frac{1}{2}}{2} = \frac{3}{8},$$

so the height of the second rectangle is

$$f\left(\frac{3}{8}\right) = \left(\frac{3}{8}\right)^3 = \frac{27}{512}.$$

The midpoint of the third interval is

$$m := \frac{\frac{1}{2} + \frac{3}{4}}{2} = \frac{5}{8},$$

so the height of the third rectangle is

$$f\left(\frac{5}{8}\right) = \left(\frac{5}{8}\right)^3 = \frac{125}{512}.$$

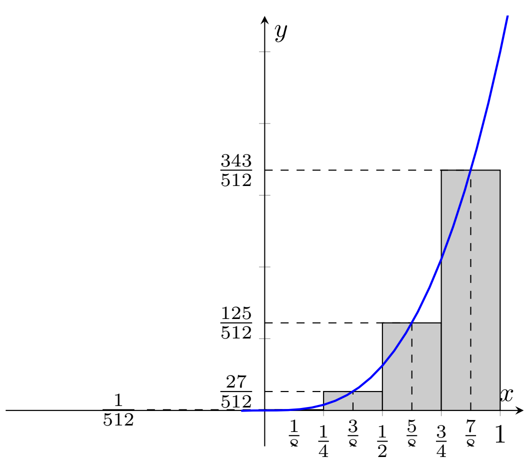
The midpoint of the fourth interval is

$$m := \frac{\frac{3}{4} + 1}{2} = \frac{7}{8},$$

so the height of the fourth rectangle is

$$f\left(\frac{7}{8}\right) = \left(\frac{7}{8}\right)^3 = \frac{343}{512}.$$

Error is $E := \left|\frac{1}{4} - \frac{7}{32}\right| = \left|\frac{8-7}{32}\right| = |1|\,32 \approx 0.03125.$
Draw the two rectangles on the graph of $y = x^3$ as shown:



The sum of the area of the rectangles is:

$$A = \frac{1}{4}\cdot\frac{1}{512} + \frac{1}{4}\cdot\frac{27}{512} + \frac{1}{4}\cdot\frac{125}{512} + \frac{1}{4}\cdot\frac{343}{512} = \frac{1+27+125+343}{4\cdot 512} = \frac{496}{4\cdot 512} = \frac{124}{512} = \frac{31}{128} \approx 0.2422.$$

Error is $E := \left|\frac{1}{4} - \frac{31}{128}\right| = \left|\frac{32-1}{128}\right| = |1|\,128 \approx 0.0078125.$

Problem 5. **[15 pt]** Write a computer program in your language of choice[2] that uses the midpoint rule to approximate the area under the graph of $f(x) = x^3$ between $x = 0$ and $x = 1$, for $n = 100$, 500, and 1000 rectangles of equal length. What is the estimate of the area in each case, and what is the error in each case? You must show the entire text of your code, and show its output.

Solution.
A short Python code that does this is:

```
#!/bin/python

n = 1000

area = 0.0
for i in range(1,n):
  midpt = (2.0*i + 1)/(2.0*n)
  area = area + midpt**3
area = area/n

print "Area is approximately "+str(area)
print "Error is approximately "+str(abs(0.25-area))
```

Running it for $n = 100$ gives the following output.

```
Area is approximately 0.24998749875
Error is approximately 1.250125e-05
```

Solution. Running it for $n = 500$ gives the following output.

```
Area is approximately 0.249999499998
Error is approximately 5.00002000026e-07
```

Solution. Running it for $n = 1000$ gives the following output.

```
Area is approximately 0.249999875
Error is approximately 1.25000124807e-07
```

[2] If you don't program yet, then I suggest MATLAB or Python, both straightforward languages that you can start with quickly. A free, in-browser Python interpreter can be found at `https://repl.it/languages/python3`, and a good introduction to scientific computing with Python can be found in the first chapter of `http://hplgit.github.io/primer.html/doc/pub/half/book.pdf` . **Please let me know early if you are stuck on this problem!**

Problem 6. Write the following sums without sigma notation, and evaluate them.

(a) **[2 points]** $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

Solution. $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k} = \tfrac{1-1}{1} + \tfrac{2-1}{2} + \tfrac{3-1}{3} = 0 + \tfrac{1}{2} + \tfrac{2}{3} = \tfrac{3}{6} + \tfrac{4}{6} = \tfrac{7}{6}.$

(b) **[2 points]** $\displaystyle\sum_{k=1}^{5} \sin(k\pi)$

Solution. $\displaystyle\sum_{k=1}^{5} \sin(k\pi) = \sin(1\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) + \sin(5\pi) = 0 + 0 + 0 + 0 + 0 = 0.$

(c) **[2 points]** $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1}$

Solution. $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1} = \tfrac{6(1)}{1+1} + \tfrac{6(2)}{2+1} = \tfrac{6}{2} + \tfrac{12}{3} = 3 + 4 = 7.$

Problem 7. Evaluate the following definite integrals.

(a) **[3 points]** $\displaystyle\int_0^2 x(x-3)\ \mathrm{d}x$

Solution.

$$\int_0^2 x(x-3)\ \mathrm{d}x = \int_0^2 x^2 - 3x\ \mathrm{d}x = \left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_0^2$$
$$= \frac{1}{3}(2)^3 - \frac{3}{2}(2)^2 - \frac{1}{3}(0)^3 + \frac{3}{2}(0)^2 = \frac{8}{3} - \frac{3\cdot 4}{2} = \frac{8}{3} - \frac{12}{2} = \frac{8}{3} - 6 = \frac{8-18}{3} = -\frac{10}{3}.$$

(b) **[3 points]** $\displaystyle\int_0^{\pi/4} \tan^2 x\ \mathrm{d}x$

Solution.

$$\int_0^{\pi/4} \tan^2 x\ \mathrm{d}x = \int_0^{\pi/4} \sec^2 x - 1\ \mathrm{d}x = [\tan x - x]_0^{\pi/4} = \tan(0) - 0 - \tan\left(\frac{\pi}{4}\right) + \frac{\pi}{4} = \frac{\pi}{4} - 1 \approx -0.214$$

(c) **[3 points]** $\displaystyle\int_0^{\ln 2} e^{3x}\ \mathrm{d}x$

Solution. $\displaystyle\int_0^{\ln 2} e^{3x}\ \mathrm{d}x = \left[\frac{1}{3}e^{3x}\right]_0^{\ln 2} = \frac{1}{3}e^{3\ln 2} - \frac{1}{3}e^0 = \frac{2^3}{3} - 1 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$