Calculus II
E1 Term, Sections E101 and E196
Instructor: E.M. Kiley
Due June 6, 2016

Week 3: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it. An example of acceptable homework solutions is posted on myWPI under "Course Materials".

Reading

Please read Section 5.5 in time for Tuesday's lecture, and Section 5.6 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 6, "Questions to Guide Your Review", p. 415, Problems 1, 2, 4, 5

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 6.1, Problems 1–49 odd
- Section 6.3, Problems 1–25 odd; 26
- Section 6.4, Problems 5–17 odd

[1]See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

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Week 6: Homework Problems

Due date: Monday, June 6, 2015, 11:59 p.m. EDT. Please upload a `.pdf` version to myWPI (`my.wpi.edu`).

Rules for Calculus Assignments:

I) Each student must compose his or her assignments independently. However, brainstorming may be done in groups.

II) Please typeset your solutions using LaTeX, or handwrite them neatly and legibly using correct English.

III) Show your work. Explain your answers using **full English sentences**.

IV) **No late assignments will be accepted for credit.**

Problem 1. **[20 points]** Use a definite integral to verify the familiar formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a right circular cone with base radius r and height h.

Solution. One way to do this is to represent the cone as the rotation of the line

$$y = \frac{r}{h}x$$

about the x-axis. (This means the cone's vertex is at the origin.) The radius is therefore

$$R(x) = \frac{rx}{h},$$

for x-values in the interval $[0, h]$. The volume of the cone is therefore

$$V = \int_0^h \pi \left(\frac{rx}{h}\right)^2 \ \mathrm{d}x = \frac{\pi r^2}{h^2} \int_0^h x^2 \ \mathrm{d}x = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2 h^3}{3h^2} - 0 = \frac{1}{3}\pi r^2 h,$$

just as desired.
You might have used a different line (for example, you might have assumed the base of the cone was centered on the origin, and the vertex was at $(h, 0)$), but the definite integral—when you choose the bounds correctly—works out to be the same (it has to, because it represents the same volume).

Problem 2. **[20 points]** A manufacturer needs to make corrugated metal sheets 36 inches wide with cross-sections in the shape of the curve $y = \frac{1}{2}\sin(\pi x)$, for $0 \le x \le 36$. How wide must the original flat sheets be, in order for the manufacturer to produce these corrugated sheets?

Solution. This question was asking you to find the arc length of the curve. Start by computing the derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\sin(\pi x)\right] = \frac{\pi}{2}\cos(\pi x),$$

so that

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{\pi^2}{4}\cos^2(\pi x),$$

and

$$1 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 - \frac{\pi^2}{4}\cos^2(\pi x) = \frac{4-\pi^2\cos^2(\pi x)}{4}.$$

Therefore, the arc length is

$$L = \int_0^{36}\sqrt{\frac{4-\pi^2\cos^2(\pi x)}{4}}\ \mathrm{d}x = \frac{1}{2}\int_0^{36}\sqrt{4-\pi^2\cos^2(\pi x)}\ \mathrm{d}x.$$

This integral is not evaluable by standard techniques, and must be approximated. I didn't expect you to do this, but the approximate value of the integral, using a left-hand Riemann sum, is 52.6. Therefore, the manufacturer needs 52.6 inches of material to make his 36-inch corrugated metal sheet.

Problem 3. The astroid shown in the figure below has equation $x^{2/3} + y^{2/3} = 1$.

(a) [**10 points**] Find the total length of the astroid.

Solution. We solve for x in terms of y for the astroid (because we will need this in part (b)). That is,

$$x^{2/3} + y^{2/3} = 1 \iff x^{2/3} = 1 - y^{2/3} \iff x = \left(1 - y^{2/3}\right)^{3/2}.$$

Therefore,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(1-y^{2/3}\right)^{3/2}\right] = \frac{3}{2}\left(1-y^{2/3}\right)^{1/2}\cdot\left(-\frac{2}{3}y^{-1/3}\right) = \frac{-\sqrt{1-y^{2/3}}}{\sqrt[3]{y}}.$$

Then

$$\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 = \left(\frac{-\sqrt{1-y^{2/3}}}{\sqrt[3]{y}}\right)^2 = \frac{1-y^{2/3}}{y^{2/3}} = \frac{1}{y^{2/3}} - 1 = y^{-2/3} - 1,$$

and

$$\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} = \sqrt{1 + y^{-2/3} - 1} = \sqrt{y^{-2/3}} = y^{-1/3}.$$

Due to symmetry, the total length of the astroid is four times the length in one quadrant, which is described when the y-values vary from 0 to 1. Therefore, the total length of the astroid is given by the formula

$$L = 4\int_0^1 \sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y = 4\int_0^1 y^{-1/3}\ \mathrm{d}y = 4\cdot\frac{3}{2}\left[y^{2/3}\right]_0^1 = 6\,[1-0] = 6.$$

(b) [**10 points**] Find the area of the surface of revolution generated by rotating the astroid around the y-axis.

Solution. We already computed $\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} = y^{-1/3}$ in the previous problem. Again due to symmetry, the total surface area of the astroid will be twice that in the half represented by rotating the portion in the first quadrant about the axis—so the formula for the surface area when revolving about the y-axis is

$$A = 2 \cdot 2\pi \int_0^1 x\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}x = 4\pi \int_0^1 \left(1-y^{2/3}\right)^{3/2} y^{-1/3}\ \mathrm{d}y.$$

This integral can be computed using the substitution $u := y^{2/3},\ \ \mathrm{d}u = \frac{2}{3}y^{-1/3}\ \mathrm{d}y,\ u(0) = 0,\ u(1) = 1.$ With this substitution, the integral transforms into

$$A = 4\pi \int_0^1 (1-u)^{3/2} \cdot \frac{3}{2}\ \mathrm{d}u = 6\pi \int_0^1 (1-u)^{3/2}\ \mathrm{d}u.$$

We apply the second substitution $v := 1-u,\ \ \mathrm{d}v = -\ \mathrm{d}u,\ v(0) = 1,\ v(1) = 0$, to obtain

$$A = -6\pi \int_1^0 v^{3/2}\ \mathrm{d}v = 6\pi \int_0^1 v^{3/2}\ \mathrm{d}v = 6\pi \cdot \frac{2}{5}\left[v^{5/2}\right]_0^1 = \frac{12\pi}{5}[1-0] = \frac{12\pi}{5}.$$