Calculus III
E2 Term, Sections E201 and E296
Instructor: E.M. Kiley
Due July 13, 2015

Week 1: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it.

Reading

Please read Sections 4.5 and 8.8 in time for Tuesday's lecture, and Section 10.1 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 4, "Questions to Guide Your Review", p. 291, Problems 17–19
- Chapter 8, "Questions to Guide Your Review", p. 529, Problems 12 and 13
- Chapter 10, "Questions to Guide Your Review", p. 647, Problems 1–5

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 4.5, Problems 1–3; 7–21 odd; 39–45 odd; 51–56 odd
- Section 8.8, Problems 1–23 odd; 35–49 odd; 65
- Section 10.1, Problems 1–25 odd; 27–51 odd

[1] See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

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Name: ______________________

WPI Username: ______________________

Week 1: Homework Problems

Due date: Sunday, July 13, 2015, 11:59 p.m. EDT. Please upload a `.pdf` version to myWPI (`my.wpi.edu`).

Rules for Calculus Assignments:

I) Each student must compose his or her assignments independently. However, brainstorming may be done in groups.

II) Please typeset your solutions using LaTeX, or handwrite them neatly and legibly using correct English.

III) Show your work. Explain your answers using **full English sentences**.

IV) No late assignments will be accepted for credit.

Problem 1. This exercise explores the difference between the limits

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x \text{ and } \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

(a) [**5 points**] Use l'Hôpital's Rule to compute the limit:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

(b) [**4 points**] On the same axes, graph the functions

$$f(x) := \left(1+\frac{1}{x}\right)^x \text{ and } g(x) := \left(1+\frac{1}{x^2}\right)^x,$$

for $x \geq 0$. How does the behavior of g compare with that of f? Use your graph and your knowledge of $\lim_{x\to\infty} f(x)$ from part (a) to estimate the value of $\lim_{x\to\infty} g(x)$.

(c) [**5 points**] Check your estimate from part (b) by using l'Hôpital's rule to compute the limit:

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x.$$

Problem 2. [**6 points**] Compute the limit: $\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^k$, where r is a real constant.

Problem 3. Euler's Gamma Function $\Gamma(x)$ uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t, \quad x > 0.$$

That is, for each positive x, the number $\Gamma(x)$ is the integral of the function $f(t) := t^{x-1}e^{-t}$ with respect to t over the interval $t \in [0,\infty)$. This definition can be extended to negative noninteger values of x by using the formula $\Gamma(x) = \frac{\Gamma(x+1)}{x}$, which we will confirm for the nonnegative integers in this exercise.

(a) [**5 points**] Show that $\Gamma(1) = 1$.

(b) [4 points] Apply integration by parts to the integral for $\Gamma(x+1)$ to show that $\Gamma(x+1) = x\Gamma(x)$. This gives the sequence:

$$\begin{aligned}\Gamma(2) &= 1\Gamma(1) = 1\\ \Gamma(2) &= 2\Gamma(2) = 2\\ \Gamma(4) &= 3\Gamma(3) = 6\\ &\vdots\\ \Gamma(n+1) &= n\Gamma(1) = n!\end{aligned}$$

(c) [1 point] Use the principle of mathematical induction[2] to show that the above sequence holds for every nonnegative integer n.

Problem 4. This problem is intended to show that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ is not necessarily equal to $\lim_{c\to\infty} \int_{-c}^{c} f(x)\,\mathrm{d}x$.

(a) [4 points] Show that $\int_0^{\infty} \frac{2x\,\mathrm{d}x}{x^2+1}$ diverges, and conclude that $\int_{-\infty}^{\infty} \frac{2x\,\mathrm{d}x}{x^2+1}$ diverges as well.

(b) [3 points] Show that $\lim_{c\to\infty} \int_{-c}^{c} \frac{2x\,\mathrm{d}x}{x^2+1} = 0$.

(c) [3 points] Using the definitions on Page 505, write a correct expression for $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$, where $f(x)$ is continuous on $(-\infty, \infty)$. Your expression should involve two different limits with two different limiting variables (do not just copy item 3 from the definition; use items 1 and 2 to expand it).

Problem 5. [10 points] Use the definition of convergence on Page 574 of the text to prove that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$. Hint: $|\sin(x)| < 1$ for all x.

Problem 6. Newton's method, applied to a differentiable function $f(x)$, begins with a starting value x_0 and generates from it a sequence of numbers $\{x_n\}$ that, under favorable circumstances, converges to a zero of f. The recursion formula for the sequence is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(a) [5 points] Show that the recursion formula for $f(x) = x^2 - a$, for $a > 0$ constant, can be written as $x_{n+1} = \frac{x_n + \frac{a}{x_n}}{2}$.

(b) [5 points] Starting with $x_0 = 1$ and $a = 3$, calculate successive terms of the sequence until the display on your calculator or computer prompt begins to repeat. What number is being approximated? Please refer to the function $f(x)$ in your answer.

[2] Once you've done parts (a) and (b), this is a very easy step; if you have never used the principle of mathematical induction before, then please look at the first section (the first five lines) of this document: `http://www.cs.cornell.edu/courses/cs2800/2014fa/handouts/toronto_induction.pdf`. The document also contains some fun mathematics problems at the end, but these are not relevant for this course.