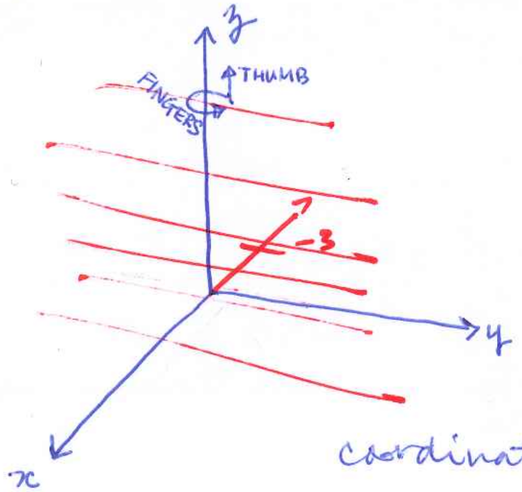
Lecture 9: 3D coordinate systems; Vectors; Products (12.1-4)

Announcements: • HW5 due Monday, 11:59 p.m.

12.1 : Three-dimensional coordinate systems.

"Right-handed" Cartesian coordinate system.



Cartesian coordinates ("rectangular coordinates") are $P(x, y, z)$.

Origin at $(0,0,0)$.

Planes determined by the coordinate axes are:
- xy-plane $(z=0)$
- yz-plane $(x=0)$
- xz-plane $(y=0)$

These planes divide three-dimensional space into eight OCTANTS, similar to 2D quadrants. The "first octant" has all coordinates (x, y, z) positive (there is no convention for numbering the other seven octants).

EXAMPLES 1, p. 705

(a) $z \geq 0$ is the half-space consisting of the points on and above the xy-plane

(b) $x = -3$ is the plane perpendicular to the x-axis (or parallel to the yz-plane) at $x = -3$.

(c) $z = 0$, $x \leq 0$ and $y \geq 0$ is the second quadrant of the xy-plane.

L9, ct'd.

EXAMPLES 1, ct'd.

(d) $\{(x,y,z) : x \geq 0,\ y \geq 0,\ z \geq 0\}$ is the first octant

(e) $\{(x,y,z) : -1 \leq y \leq 1\}$ is the slab between (including) the planes $y=-1$ and $y=1$.

(f) $\{(x,y,z) : y=-2,\ z=2\}$ is the line in which the two planes $y=-2$ and $z=2$ intersect. Also, the line through the point $(0,-2,2)$ parallel to the x-axis.

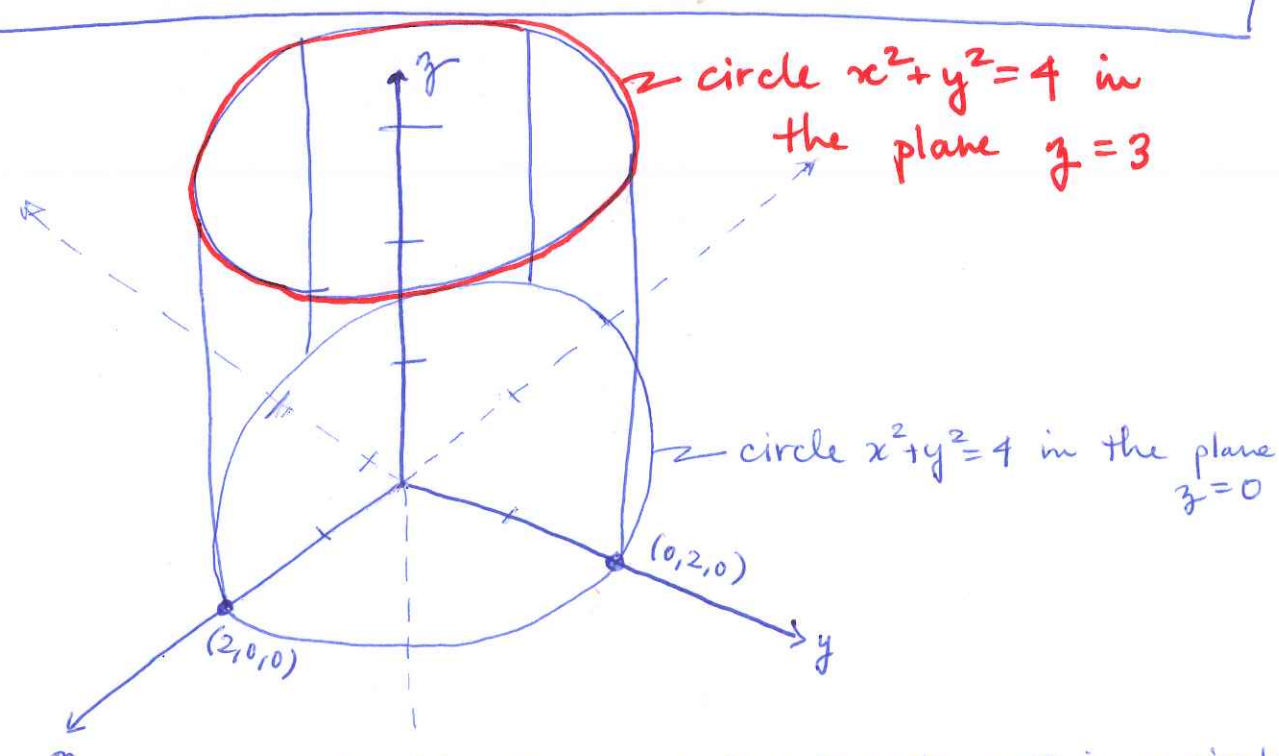
EXAMPLE 2, p. 705

Which points (x,y,z) satisfy the eq'ns

$$x^2+y^2=4 \quad \text{and} \quad z=3 \ ?$$

We know the points lie in the horizontal plane $z=3$, and also that $x^2+y^2=4$ — i.e., they lie on the

$$\boxed{\text{circle centered at } (0,0,3) \text{ with radius } 2 \text{ in the plane } z=3.}$$



L9, ct'd.

Distance in space.

Recall: in 2D, the distance btwn. (x_1, y_1) and (x_2, y_2) is:

$$\text{dist} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

In 3D, very similar – the dist. btwn (x_1, y_1, z_1) & (x_2, y_2, z_2):

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}}$$

(can prove using Pythagorean theorem)

EXAMPLE 3, p. 706

The distance btwn. $P_1(2,1,5)$ and $P_2(-2,3,0)$:

$$|P_1P_2| = \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2}$$

$$= \sqrt{(-4)^2 + (2)^2 + (-5)^2}$$

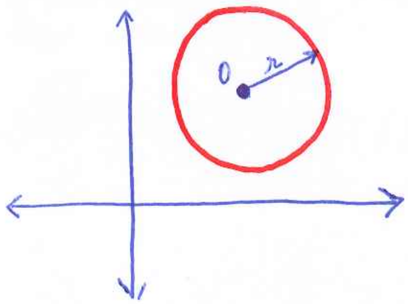
$$= \sqrt{16+4+25}$$

$$= \sqrt{45} = \sqrt{9 \cdot 5} = \boxed{3\sqrt{5} \approx 6.708.}$$

L9, ct'd.

Spheres in space

Q. How to define a circle in 2D?

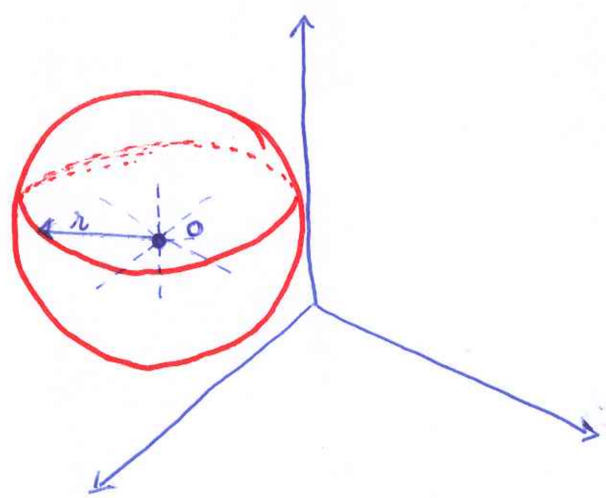


$$\text{circle}(0, r) = \{(x,y) : |0,(x,y)| = r\}$$

$$= \{(x,y) : \underbrace{(x-a)^2 + (y-b)^2 = r^2}_{\text{from dist. formula in 2D}}\}$$

So... how to define a sphere in 3D?

the set of points $P(x,y,z)$ that are exactly distance r away from the center $O(a,b,c)$:



$$\text{sphere}(0, r) = \{(x,y,z) : |0,(x,y,z)| = r\}$$

$$\boxed{= \{(x,y,z) : (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2\}}$$

EXAMPLE
4, p. 706

Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Strategy: complete the square in x, in y, and in z.

$$(x+a)^2 = x^2 + 2ax + a$$

$$[x^2 + \underbrace{3x}_{\text{add } \left(\frac{3}{2}\right)^2 \text{ to both sides}}] + [y^2] + [z^2 \underbrace{- 4z}_{\text{add } \left(\frac{-4}{2}\right)^2 \text{ to both sides}}] = -1$$

EXAMPLE 4, ct'd.

$$\left[x^2+3x+\left(\tfrac{3}{2}\right)^2\right]+\left[y^2\right]+\left[z^2-4z+\left(\tfrac{-4}{2}\right)^2\right]=-1+\left(\tfrac{3}{2}\right)^2+\left(\tfrac{-4}{2}\right)^2$$

$$\left(x+\tfrac{3}{2}\right)^2+(y+0)^2+\left(z-\cancel{\tfrac{4}{2}}\,2\right)^2=-1+\frac{9}{4}+4=3+\frac{9}{4}=\frac{21}{4}$$

So, radius is $\sqrt{\frac{21}{4}}=\frac{\sqrt{21}}{2}$, ctr. is $(-3/2,\,0,\,2)$.

EXAMPLE 5, p.707 Some geometric interpretations of (inequalities / eq'ns) involving spheres:

(a) $\{(x,y,z): x^2+y^2+z^2<4\}$ all points inside the shell $x^2+y^2+z^2=4$.

(b) $\{(x,y,z): x^2+y^2+z^2\le 4\}$ the solid sphere $x^2+y^2+z^2=4$ and its interior

(c) $\{(x,y,z): x^2+y^2+z^2>4\}$ the exterior of the shell $x^2+y^2+z^2=4$

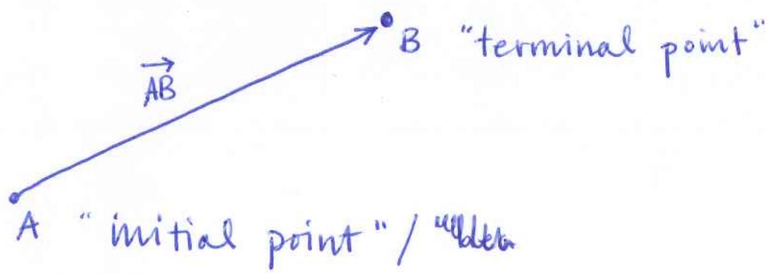
(d) $\{(x,y,z): x^2+y^2+z^2=4,\ z\le 0\}$ the lower hemisphere of the sphere $x^2+y^2+z^2=4$ cut by the plane $z=0$. Note: the plane $z=0$ is not part of this set!

L9, ct'd.

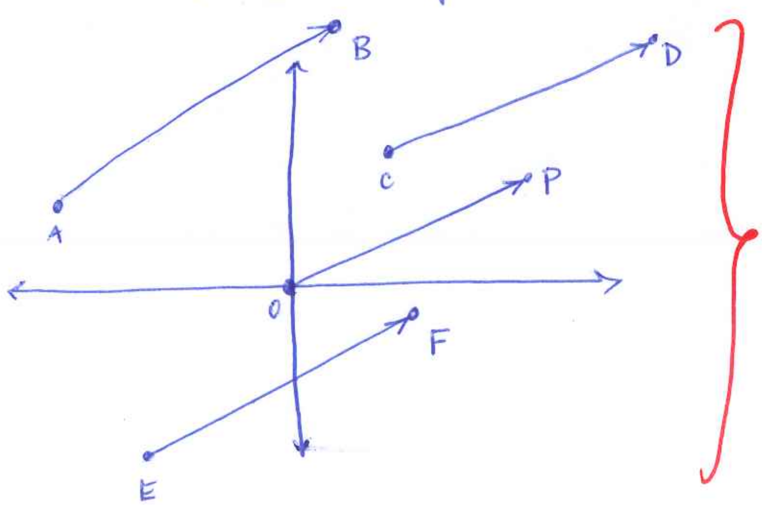
12.2 : Vectors.

- Describes quantities with MAGNITUDE & DIRECTION, e.g.
  - Force
  - Displacement
  - Velocity

In $\mathbb{R}^3$ the 3D Cartesian space, a vector is represented by a directed line segment :



- Two vectors are equal if they have the same magnitude and direction, REGARDLESS of the initial point :



$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}$

Typically, the vector w/starting (initial) point at the origin is said to be in standard form and is the vector

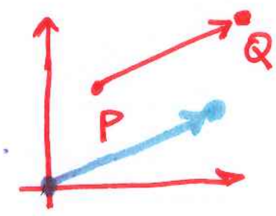
- When working with standard vectors, we represent them by giving the coordinates of their terminal points:

> **DEF.** If $\vec{v}$ is a 2D vector in the plane equal to the vector with initial point at the origin and terminal point at (v_1, v_2), then the COMPONENT FORM of $\vec{v}$:
>
> $$\vec{v} = \langle v_1, v_2 \rangle.$$
>
> In 3D, the analogue is
>
> $$\vec{v} = \langle v_1, v_2, v_3 \rangle.$$

- We call v_1, v_2, and v_3 the COMPONENTS of $\vec{v}$.

Q: How to compute a standard vector from a general vector w/initial pt. $P(x_1, y_1, z_1)$ and terminal $Q(x_2, y_2, z_2)$

A: $$\vec{PQ} = \vec{v} := \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$



Why? - subtract P from Q to find "new" terminal pt.

Length of vectors.

DEF. The magnitude or length of the vector $\vec{v} = \vec{PQ}$:

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad \Big(= \underbrace{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}}_{\text{distance formula in 3D}} \Big)$$

• The only vector w/ length 0 is the ZERO VECTOR $\langle 0,0,0 \rangle$, written $\vec{0}$. (In 2D, $\vec{0} := \langle 0,0 \rangle$.) The zero vector also has no specific direc'n.

EXAMPLE 1, p. 710 Find (a) component form & (b) length of the vector $\vec{PQ}$, $P(-3,4,1)$ and $Q(-5,2,2)$.

(a) $\vec{v} := Q - P = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$

$= \langle -5-(-3),\ 2-4,\ 2-1 \rangle = \langle -2, -2, 1 \rangle$.

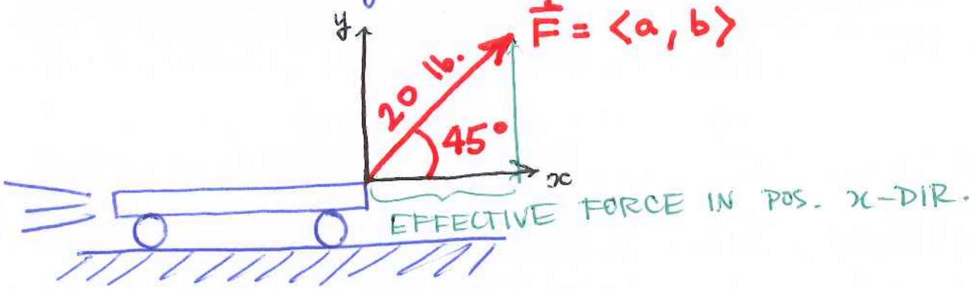
(b) $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

$= \sqrt{(-2)^2 + (-2)^2 + (1)^2}$

$= \sqrt{4+4+1}$

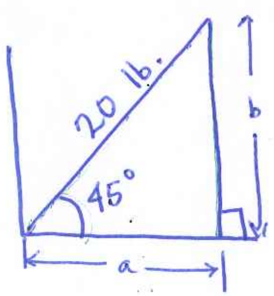
$= \sqrt{9} = 3$

EXAMPLE 2, p. 711 A small cart is being pulled along a smooth horizontal floor with a 20-lb. force $\vec{F}$ making a 45° angle to the floor. What is the effective force moving the cart forward?



The effective force moving the cart forward is just the x-component of $\vec{F}$. In our diagram, that's a.

So look:



Recall trig. formulas for the right triangle:

$$\sin(\theta) = \frac{\text{OPPOSITE}}{\text{HYPOTENUSE}}, \quad \cos(\theta) = \frac{\text{ADJACENT}}{\text{HYPOTENUSE}}$$

"$S\frac{O}{H}$ $C\frac{A}{H}$ $T\frac{O}{A}$"

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\text{OPPOSITE}}{\text{ADJACENT}}$$

Then [redacted] $\cos(45°) = \dfrac{a}{20\text{ lb}}$ implies

$$a = (20\text{ lb})\cos(45°) = 20\left(\frac{\sqrt{2}}{2}\right)^{\text{lb}} = 10\sqrt{2}\text{ lb} \approx \boxed{14.14\text{ lb}}$$

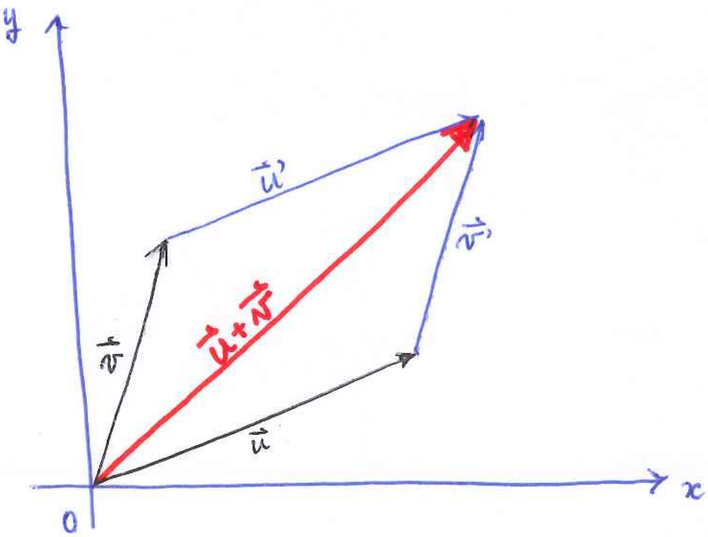
Vector algebra.

DEF. Let $\vec{u} := \langle u_1, u_2, u_3 \rangle$ and $\vec{v} := \langle v_1, v_2, v_3 \rangle$ be vectors, and let k be a scalar (a real #). Then:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

$$k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$$

- "Head-to-tail" addition:



- What is the length of a scalar multiple of $\vec{u}$?

$$|k\vec{u}| = |\langle ku_1, ku_2, ku_3 \rangle| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} =$$

$$= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} =$$

$$= |k|\sqrt{u_1^2 + u_2^2 + u_3^2} = |k||\vec{u}|$$

this |·| is vector length

this |·| is absolute value (of scalar

L9, ct'd.

EXAMPLE
3, p. 712

Let $\vec{u} := \langle -1, 3, 1\rangle$ and $\vec{v} := \langle 4, 7, 0\rangle$. Find:

(a) $2\vec{u} + 3\vec{v}$; $2\vec{u} = \langle 2(-1), 2(3), 2(1)\rangle$

$= \langle -2, 6, 2\rangle$

$3\vec{v} = \langle 3(4), 3(7), 3(0)\rangle$

$= \langle 12, 21, 0\rangle$

$\therefore\ 2\vec{u} + 3\vec{v} = \langle -2+12, 6+21, 2+0\rangle$

$= \langle 10, 27, 2\rangle$

(b) $\vec{u} - \vec{v} = \langle -1-4, 3-7, 1-0\rangle$

$= \langle -5, -4, 1\rangle$

(c) $\left|\frac{1}{2}\vec{u}\right| = \left|\frac{1}{2}\langle -1, 3, 1\rangle\right| = \left|\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\rangle\right| =$

$= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}}$

$= \sqrt{11/4} = \frac{1}{2}\sqrt{11}$.

Alt. $|\vec{u}| = |\langle -1, 3, 1\rangle| = \sqrt{1+9+1} = \sqrt{11}$

$\left|\frac{1}{2}\vec{u}\right| = \left|\frac{1}{2}\right| |\vec{u}| = \frac{1}{2}\sqrt{11}$.

L9, ct'd.

- We have no reason (yet) to believe that vector add'n and scalar multiplication have the same nice properties that scalar addition and multiplication of two scalars have. (e.g. – commutativity, associativity, etc.) But let's check:

Let $\vec{u} := \langle u_1, u_2, u_3 \rangle$, $\vec{v} := \langle v_1, v_2, v_3 \rangle$, $\vec{w} := \langle w_1, w_2, w_3 \rangle$

and let $a, b \in \mathbb{R}$.

(1) $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

$= \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$

$= \vec{v} + \vec{u}$.

because $\boxed{\text{scalar add'n}}$ is commutative!

So vector add'n is commutative.

(2) $(\vec{u} + \vec{v}) + \vec{w} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle + \vec{w}$

$= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3 \rangle$

because scalar add'n is ~~distributive~~ associative

$= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3) \rangle$

$= \vec{u} + \langle v_1 + w_1, v_2 + w_2 + v_3 + w_3 \rangle$

$= \vec{u} + (\vec{v} + \vec{w})$.

So vector add'n is ~~distributive~~ associative.

Properties of vector operations.

(1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(3) $\vec{u} + \vec{0} = \vec{u}$

(4) $\vec{u} + (-\vec{u}) = \vec{0}$

(5) $0\vec{u} = \vec{0}$

(6) $1\vec{u} = \vec{u}$

(7) $a(b\vec{u}) = (ab)\vec{u}$

vector add'n

(8) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

scalar

(9) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

Can prove all of these!

Unit vectors

A vector $\vec{v}$ whose length is 1 is a <u>UNIT VECTOR</u>. $(\hat{v})$

The <u>STANDARD UNIT VECTORS</u> are:

$$\hat{\imath} := \langle 1, 0, 0 \rangle$$
$$\hat{\jmath} := \langle 0, 1, 0 \rangle$$
$$\hat{k} := \langle 0, 0, 1 \rangle .$$

(There are other unit vectors, of course.)

Any vector $\vec{u}$ can be written as a <u>LINEAR COMBINATION</u> of the standard unit vectors:

$$\begin{aligned}
\vec{u} &= \langle u_1, u_2, u_3 \rangle \\
&= \langle u_1, 0, 0 \rangle + \langle 0, u_2, 0 \rangle + \langle 0, 0, u_3 \rangle \\
&= u_1 \langle 1, 0, 0 \rangle + u_2 \langle 0, 1, 0 \rangle + u_3 \langle 0, 0, 1 \rangle \\
&= \underbrace{u_1}_{\text{"i-component"}} \hat{\imath} + \underbrace{u_2}_{\text{"j-component"}} \hat{\jmath} + \underbrace{u_3}_{\text{"k-component"}} \hat{k} .
\end{aligned}$$

Unit vectors in the direc'n of other vectors.

Whenever $\vec{v} \neq \vec{0}$, its length $|\vec{v}| \neq 0$ (why?),

so $$\underbrace{\left|\frac{\vec{v}}{|\vec{v}|}\right|}_{\text{vec. length}} = \underbrace{\left|\frac{1}{\underbrace{|\vec{v}|}_{\text{length}}}\right|}_{\text{abs. val}} |\vec{v}| = \frac{1}{|\vec{v}|} |\vec{v}| = 1,$$

and therefore $\frac{\vec{v}}{|\vec{v}|}$ is a unit vector in the same direction as $\vec{v}$.

EXAMPLE 4, p. 713

Find a unit vector $\hat{u}$ in the same direc'n as the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$.

First, get $\overline{P_1P_2}$ in standard form:

$$\vec{v} := P_2 - P_1 = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$
$$= \langle 3-1, 2-0, 0-1 \rangle = \langle 2, 2, -1 \rangle.$$

Then compute $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{2^2 + 2^2 + (-1)^2} =$
$$= \sqrt{4+4+1} = \sqrt{9} = 3$$

And set $$\hat{u} := \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 2,2,-1 \rangle}{3} = \boxed{\left\langle \frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \right\rangle}.$$

L9, ct'd.

EXAMPLE
5, p. 713

If $\vec{v} \doteq 3\hat{\imath} - 4\hat{\jmath}$ is a velocity vector, express $\vec{v}$ as a product of its speed and its direc'n of motion.

Well, speed is the magnitude of velocity:

$$s \doteq |\vec{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5,$$

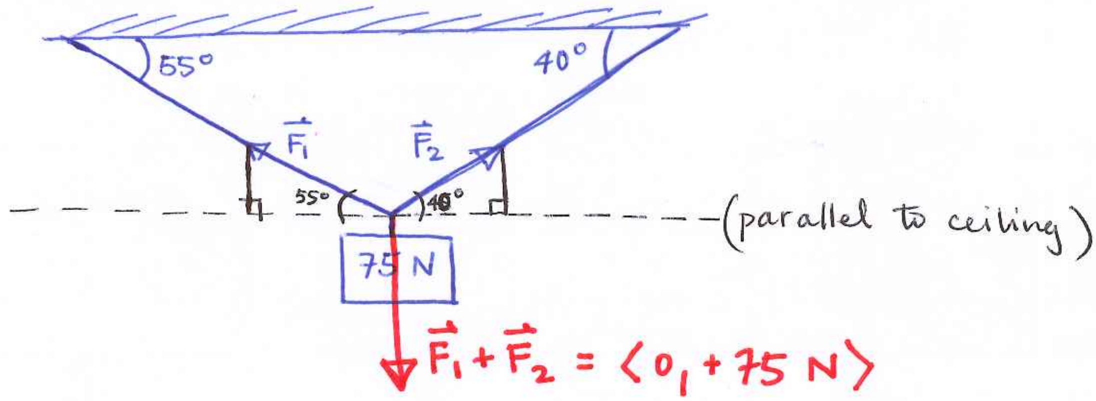
and the direc'n is expressed as a unit vector:

$$\hat{u} := \frac{\vec{v}}{|\vec{v}|} = \frac{3\hat{\imath} - 4\hat{\jmath}}{5} = \frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath},$$

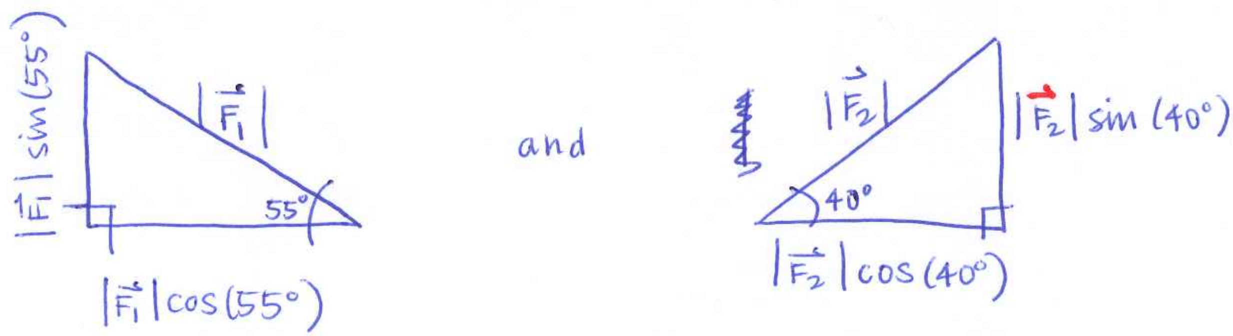
and therefore,

$$\vec{v} = s\hat{u} = \boxed{5\left(\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}\right)}.$$

EXAMPLE 9, p. 715 A 75-N weight is suspended by two wires as shown. Find the forces $\vec{F_1}$ and $\vec{F_2}$ acting in both wires.



Note:



So
$$\vec{F_1} = \langle \boxed{-|\vec{F_1}|\cos(55°)}, \boxed{|\vec{F_1}|\sin(55°)} \rangle$$
$$+\ \vec{F_2} = \langle \boxed{|\vec{F_2}|\cos(40°)}, \boxed{|\vec{F_2}|\sin(40°)} \rangle$$

and $\vec{F_1} + \vec{F_2} = \langle 0, 75\text{ N} \rangle$ implies

$$-|\vec{F_1}|\cos(55°) + |\vec{F_2}|\cos(40°) = 0\text{ N}$$

and $|\vec{F_1}|\sin(55°) + |\vec{F_2}|\sin(40°) = 75\text{ N}.$

EXAMPLE
9, p. 715
ct'd

Solve simultaneously:

$$(1) \quad -|\vec{F_1}|\cos(55°) + |\vec{F_2}|\cos(40°) = 0\text{ N}$$

$$(2) \quad |\vec{F_1}|\sin(55°) + |\vec{F_2}|\sin(40°) = 75\text{ N}$$

Eq'n (1): $|\vec{F_1}| = |\vec{F_2}|\dfrac{\cos(40°)}{\cos(55°)}$

plug into eq'n (2):

$$|\vec{F_2}|\left(\frac{\cos(40°)\sin(55°)}{\cos(55°)} + \sin(40°)\right) = 75\text{ N}$$

$$|\vec{F_2}|\left(\cos(40°)\tan(55°) + \sin(40°)\right) = 75\text{ N}$$

$$\Rightarrow \quad |\vec{F_2}| = \frac{75\text{ N}}{\cos(40°)\tan(55°) + \sin(40°)}$$

$$\approx 43.18\text{ N}$$

$$\Rightarrow \quad |\vec{F_1}| = |\vec{F_2}|\frac{\cos(40°)}{\cos(55°)} \approx (43.18\text{ N})\frac{\cos(40°)}{\cos(55°)}$$

$$\approx 57.67\text{ N}.$$

Finally,

$$\boxed{\vec{F_1} = |\vec{F_1}|\langle -\cos(55°), \sin(55°)\rangle \approx \langle -33.08, 47.24\rangle}$$

$$\boxed{\vec{F_2}} = |\vec{F_2}|\langle \cos(40°), \sin(40°)\rangle$$

12.3: The dot product.

DEF. $\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + u_3 v_3$.

scalars

THIS IS A SCALAR.

THE DOT PRODUCT OF TWO VECTORS IS A SCALAR.

(THIS IS WHY THE DOT PRODUCT IS SOMETIMES CALLED THE SCALAR PRODUCT.)

EXAMPLE 1, p. 719

$$\langle 1, -2, 1 \rangle \cdot \langle -6, 2, -3 \rangle = 1(-6) + (-2)2 + 1(-3)$$
$$= -6 - 4 - 3$$
$$= -13.$$

$$\left(\tfrac{1}{2}\hat{\imath} + 3\hat{\jmath} + \hat{k}\right) \cdot \left(4\hat{\imath} - \hat{\jmath} + 2\hat{k}\right) = \left(\tfrac{1}{2}\right)4 + 3(-1) + 1(2)$$
$$= 2 - 3 + 2$$
$$= 1.$$

The angle btwn. two vectors $\vec{u}$, $\vec{v}$ (both nonzero) is

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$$

(derivatn, p. 718-9)

EXAMPLE 2, p.720 Find the angle btwn. $\vec{u} := \hat{\imath} - 2\hat{\jmath} - 2\hat{k}$ and $\vec{v} := 6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$.

Well, $\vec{u}\cdot\vec{v} = 1(6) + (-2)3 + (-2)2$

$= 6 - 6 - 4 = -4$

and $|\vec{u}| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

and $|\vec{v}| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{36+9+4} = \sqrt{49} = 7$

so $\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \cos^{-1}\left(\frac{-4}{3\cdot 7}\right) = \cos^{-1}\left(\frac{-4}{21}\right)$ #

$\approx$ 1.76 radians = 100.98°

DEF. Two vectors are <u>ORTHOGONAL</u> if their dot prod. is 0.

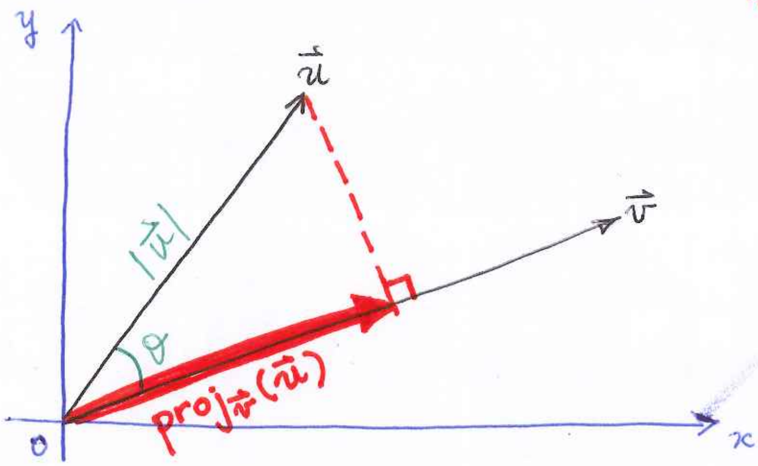
(Why? $\vec{u}\cdot\vec{v} = 0$ implies $\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \arccos(0)$

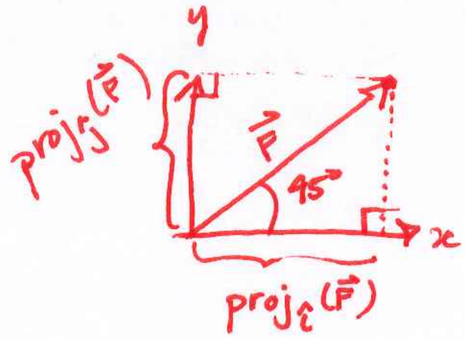
$= \frac{\pi}{2} = 90°$.)

EXAMPLE 4, p.720 Are $\vec{u} := \langle 3,-2\rangle$ and $\vec{v} := \langle 4,6\rangle$ orthogonal? YES

well, $\vec{u}\cdot\vec{v} = 3(4) + (-2)(6) = 12 + (-12) = 0$.

Vector projection.





The VECTOR PROJEC'N of $\vec{u}$ ONTO $\vec{v}$ $(\neq \vec{0})$ is the vector determined by dropping a perpendicular line segment from the head of $\vec{u}$ onto $\vec{v}$.

- If $\vec{u}$ represents a force, then $\text{proj}_{\vec{v}}(\vec{u})$ represents the EFFECTIVE force in the direction of $\vec{v}$.

$$\boxed{\begin{aligned}\text{proj}_{\vec{v}}(\vec{u}) &= \underbrace{\left(|\vec{u}|\cos\theta\right)}_{\text{magnitude}}\underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{direc'n}} \\ &= \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)}_{\text{SCALAR}}\underbrace{\vec{v}}_{\text{VECTOR}}\end{aligned}}$$

EXAMPLE 5, p. 722

Find the vector projec'n of $\vec{u} := 6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ onto $\vec{v} := \hat{\imath} - 2\hat{\jmath} - 2\hat{k}$, and find its magnitude.

$$\text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}.$$

$$\vec{u}\cdot\vec{v} = 6(1) + 3(-2) + 2(-2) = 6 - 6 - 4 = -4$$

$$|\vec{v}|^2 = 1^2 + (-2)^2 + (-2)^2 = 1 + 4 + 4 = 9$$

So $\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2} = \dfrac{-4}{9}$

and $\text{proj}_{\vec{v}}(\vec{u}) = \left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left(\dfrac{-4}{9}\right)\langle 1, -2, -2\rangle = \left\langle \dfrac{-4}{9}, \dfrac{8}{9}, \dfrac{8}{9}\right\rangle$

Now, magnitude $= |\vec{u}|\cos\theta = \cancel{|\vec{u}|}\cdot\dfrac{\vec{u}\cdot\vec{v}}{\cancel{|\vec{u}|}\,|\vec{v}|} = \dfrac{1}{|\vec{v}|}\left|(\vec{u}\cdot\vec{v})\right|$

$$= \frac{1}{\sqrt{9}}\left|(-4)\right| = \frac{4}{3}$$

$$\left|\left\langle \frac{-4}{9}, \frac{8}{9}, \frac{8}{9}\right\rangle\right| = \sqrt{\left(\frac{-4}{9}\right)^2 + \left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^2}$$

$$= \frac{1}{9}\sqrt{16 + 2(64)} = \frac{1}{9}\sqrt{16+128} = \frac{\sqrt{144}}{9}$$

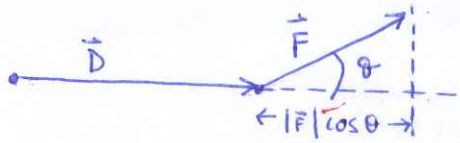
Work.

DEF. The WORK done by a constant force $\vec{F}$ acting through a displacement $\vec{D}$ is

$$W = \vec{F} \cdot \vec{D}.$$

EXAMPLE 7, p. 723

If $|\vec{F}| = 40\text{ N}$, $|\vec{D}| = 3\text{ m}$, and $\theta = 60°$, compute the work done by $\vec{F}$ in acting through $\vec{D}$.



$$\begin{aligned} \text{Work} &= \vec{F} \cdot \vec{D} \\ &= |\vec{F}||\vec{D}| \cos(\theta) \\ &= (40\text{ N})(3\text{ m}) \cos(60°) \\ &= (40\text{ N})(3\text{ m})(1/2) \\ &= 60\text{ N}\cdot\text{m} = 60\text{ J}. \end{aligned}$$

L9, ct'd.

12.4: the cross product.

DEF. The CROSS PRODUCT $\vec{u} \times \vec{v}$ is the vector

$$\vec{u} \times \vec{v} := \left(|\vec{u}||\vec{v}| \sin\theta \right) \hat{n}$$

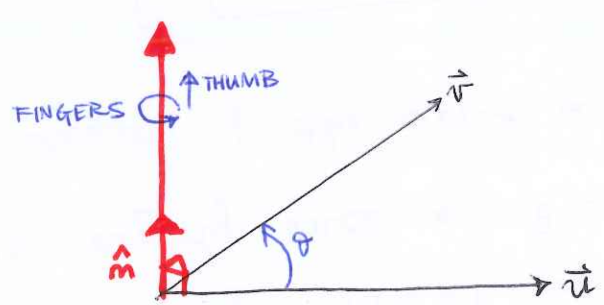
where $\hat{n}$ is the (unit) vector normal to both $\vec{u}$ and $\vec{v}$, chosen by the right-hand rule.

THIS IS A VECTOR.

THE CROSS PRODUCT OF TWO VECTORS IS A VECTOR.

(THIS IS WHY THE CROSS PRODUCT IS SOMETIMES CALLED THE **VECTOR PRODUCT**.)

The normal vector $\hat{n}$:



Note: $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

The "standard" properties (commutativity, distributivi etc.) **DO NOT ALWAYS HOLD FOR CROSS PROD.**

L9, ct'd.

The cross product can be computed as the DETERMINANT of a symbolic matrix.

Recall: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

$$\det\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right) = a_1 \det\left(\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}\right) - a_2 \det\left(\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}\right) + a_3 \det\left(\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}\right)$$

"cofactor expansion"

The cross product as a determinant:

$$\vec{u} \times \vec{v} = \det\left(\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\right)$$

$$= (u_2 v_3 - u_3 v_2)\,\hat{\imath} + (u_3 v_1 - u_1 v_3)\,\hat{\jmath} + (u_1 v_2 - u_2 v_1)\,\hat{k}$$

EXAMPLE
1, p. 728

Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ for $\vec{v} = \langle -4, 3, 1 \rangle$, $\vec{u} = \langle 2, 1, 1 \rangle$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \hat{k}$$

$$= (1-3)\,\hat{\imath} - (2+4)\,\hat{\jmath} + (6+4)\,\hat{k} = \langle -2, -6, 1$$

$\boxed{\text{EXAMPLE 2, p. 728}}$ Find a vector perpendicular to the plane that contains the points $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

Well, two vectors in this plane are $\overrightarrow{PQ}$ and $\overrightarrow{PR}$:

$$\overrightarrow{PQ} = Q-P = \langle 2-1,\ 1-(-1),\ -1-0\rangle = \langle +1, 2, -1\rangle$$

$$\overrightarrow{PR} = R-P = \langle -1-1,\ 1-(-1),\ 2-0\rangle = \langle -2, 2, 2\rangle$$

And a vector normal to both vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ will be perpendicular to the plane containing P, Q, & R.

Such a vector is exactly $\overrightarrow{PQ} \times \overrightarrow{PR}$:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\hat{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\hat{k}$$

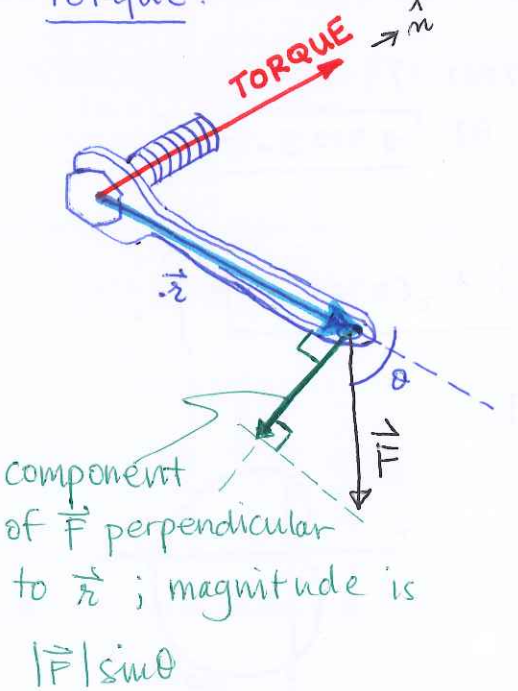
$$= (4-(-2))\hat{i} - (2-2)\hat{j} + (2-(-4))\hat{k}$$

$$= 6\hat{i} + 0\hat{j} + 6\hat{k}.$$

$\boxed{\text{EXAMPLE 4, p. 728}}$ Find a <u>unit vector</u> $\perp$ to the plane P, Q, R.

$$\hat{u} := \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{\langle 6,0,6\rangle}{|\langle 6,0,6\rangle|} = \frac{\langle 6,0,6\rangle}{\sqrt{6^2+6^2}} = \frac{\langle 6,0,6\rangle}{6\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle.$$

Torque.



TORQUE VECTOR when we use $\vec{F}$ force to turn a bolt, applied $\vec{r}$ away from the bolt's head, is:

$$\boxed{\begin{aligned}\vec{T} &= \vec{r} \times \vec{F} \\ &= \left(|\vec{r}||\vec{F}|\sin\theta\right)\hat{n}\end{aligned}}$$

Note that the magnitude of $\vec{T}$ is $|\vec{r}||\vec{F}|\sin\theta$ — so we can increase the torque by:

- Increasing $|\vec{F}|$
- Increasing $|\vec{r}|$
- Trying to get $\sin(\theta) = 1$, i.e., $\theta = \frac{\pi}{2}$.

(Aside: ever stick a long pipe over the end of a wrench to help move a stuck bolt? — — that's item (2) in the list!)