Calculus III
E2 Term, Sections E201 and E296
Instructor: E.M. Kiley
Due July 13, 2015

Week 1: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it.

Reading

Please read Sections 4.5 and 8.8 in time for Tuesday's lecture, and Section 10.1 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 4, "Questions to Guide Your Review", p. 291, Problems 17–19
- Chapter 8, "Questions to Guide Your Review", p. 529, Problems 12 and 13
- Chapter 10, "Questions to Guide Your Review", p. 647, Problems 1–5

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 4.5, Problems 1–3; 7–21 odd; 39–45 odd; 51–56 odd
- Section 8.8, Problems 1–23 odd; 35–49 odd; 65
- Section 10.1, Problems 1–25 odd; 27–51 odd

[1] See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

Calculus III
E2 Term, Sections E201 and E296
Instructor: E.M. Kiley
Due July 13, 2015

Name: ____________________

WPI Username: ____________________

Week 1: Homework Problems

Due date: Sunday, July 13, 2015, 11:59 p.m. EDT. Please upload a `.pdf` version to myWPI (`my.wpi.edu`).

Rules for Calculus Assignments:

I) Each student must compose his or her assignments independently. However, brainstorming may be done in groups.

II) Please typeset your solutions using LaTeX, or handwrite them neatly and legibly using correct English.

III) Show your work. Explain your answers using **full English sentences**.

IV) **No late assignments will be accepted for credit.**

Problem 1. This exercise explores the difference between the limits

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x \text{ and } \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

(a) [**5 points**] Use l'Hôpital's Rule to compute the limit:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

Solution.

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x &= \exp\left[\ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]\right] \\
&= \exp\left[\lim_{x\to\infty}\ln\left[\left(1+\frac{1}{x}\right)^x\right]\right], \text{ since the } \ln \text{ function is continuous} \\
&= \exp\left[\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)\right], \text{ since } \ln(a^b)=b\ln a \\
&= \exp\left[\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\right].
\end{aligned}$$

Now, we see that this limit has the "$\frac{0}{0}$" form, and so we check whether the functions $y_1(x) := \ln\left(1+\frac{1}{x}\right)$ and $y_2(x) := 1/x$ satisfy the other two conditions of l'Hôpital's rule. Well, $y_1'(x) = -\frac{1}{x^2}\frac{1}{1+\frac{1}{x}}$, and $y_2'(x) = -\frac{1}{x^2}$, and both of these exist wherever $x\neq 0$ and $x\neq -1$. Also in particular, $y_2'(x)\neq 0$, and so all conditions are satisfied, and we may apply l'Hôpital's rule:

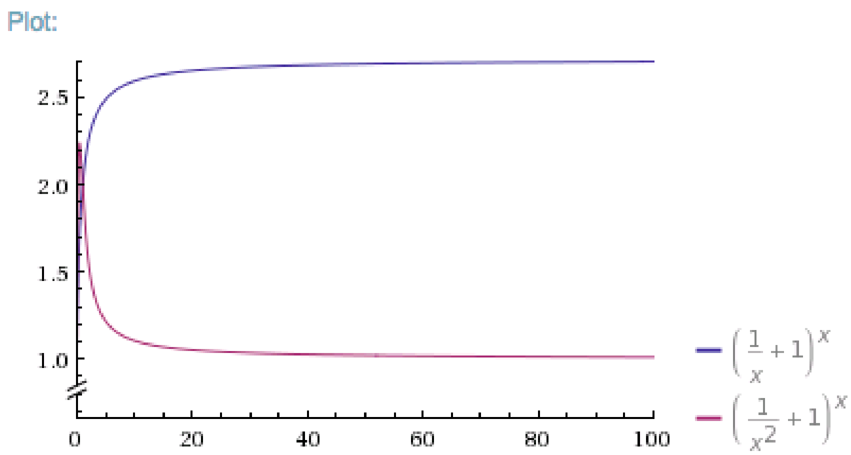
$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x &= \exp\left[\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\right] \\
&= \exp\left[\lim_{x\to\infty}\frac{\left(-1/x^2\right)\left(1/\left(1+\frac{1}{x}\right)\right)}{-1/x^2}\right] \\
&= \exp\left[\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}\right] \\
&= \exp[1] = e.
\end{aligned}$$

(b) **[4 points]** On the same axes, graph the functions

$$f(x) := \left(1 + \frac{1}{x}\right)^x \text{ and } g(x) := \left(1 + \frac{1}{x^2}\right)^x,$$

for $x \geq 0$. How does the behavior of g compare with that of f? Use your graph and your knowledge of $\lim_{x\to\infty} f(x)$ from part (a) to estimate the value of $\lim_{x\to\infty} g(x)$.

Solution. Your graph should look something like this one, generated by WolframAlpha:



We see from the graph that $f(x) \to e$ (if our graph of f didn't show a horizontal asymptote at the value we found in part (a), then that should have been a reality check!). We also see that the graph of g has a horizontal asymptote, and it looks to be about a third of the way between e and the x-axis—so $g(x) \to 1$ is a good guess.

(c) [5 points] Check your estimate from part (b) by using l'Hôpital's rule to compute the limit:

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x.$$

Solution.

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x &= \exp\left[\ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x\right]\right] \\
&= \exp\left[\lim_{x\to\infty}\ln\left[\left(1+\frac{1}{x^2}\right)^x\right]\right], \text{ since the } \ln \text{ function is continuous} \\
&= \exp\left[\lim_{x\to\infty} x\ln\left(1+\frac{1}{x^2}\right)\right], \text{ since } \ln(a^b) = b\ln a \\
&= \exp\left[\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x^2}\right)}{1/x}\right].
\end{aligned}$$

Now, we see that this limit has the "$\frac{0}{0}$" form, and so we check whether the functions $y_1(x) := \ln\left(1+\frac{1}{x^2}\right)$ and $y_2(x) := 1/x$ satisfy the other two conditions of l'Hôpital's rule. Well, $y_1'(x) = -\frac{2}{x^3}\frac{1}{1+\frac{1}{x^2}}$, and $y_2'(x) = -\frac{1}{x^2}$, and both of these exist wherever $x \neq 0$ and $x \neq -1$. Also in particular, $y_2'(x) \neq 0$, and so all conditions are satisfied, and we may apply l'Hôpital's rule:

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x &= \exp\left[\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x^2}\right)}{1/x}\right] \\
&= \exp\left[\lim_{x\to\infty}\frac{\left(-2/x^3\right)\left(1/\left(1+\frac{1}{x^2}\right)\right)}{-1/x^2}\right] \\
&= \exp\left[\lim_{x\to\infty}\frac{2}{x+\frac{1}{x^3}}\right] \\
&= exp[0] = 1.
\end{aligned}$$

So, indeed, our estimate from part (b) was accurate.

Problem 2. **[6 points]** Compute the limit: $\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^k$, where r is a real constant.

Solution.

$$\begin{aligned}
\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^k &= \exp\left[\ln\left[\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^k\right]\right] \\
&= \exp\left[\lim_{k\to\infty}\ln\left[\left(1+\frac{r}{k}\right)^k\right]\right], \text{ since the } \ln \text{ function is continuous} \\
&= \exp\left[\lim_{k\to\infty} x\ln\left(1+\frac{r}{k}\right)\right], \text{ since } \ln(a^b)=b\ln a \\
&= \exp\left[\lim_{k\to\infty}\frac{\ln\left(1+\frac{r}{k}\right)}{1/k}\right].
\end{aligned}$$

Now, we see that this limit has the "$\frac{0}{0}$" form, and so we check whether the functions $y_1(k) := \ln\left(1+\frac{r}{k}\right)$ and $y_2(k) := 1/k$ satisfy the other two conditions of l'Hôpital's rule. Well, $y_1'(x) = -\frac{r}{k^2}\frac{1}{1+\frac{r}{k}}$, and $y_2'(k) = -\frac{1}{k^2}$, and both of these exist wherever $k \neq 0$ and $k \neq -1$. Also in particular, $y_2'(k) \neq 0$, and so all conditions are satisfied, and we may apply l'Hôpital's rule:

$$\begin{aligned}
\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^k &= \exp\left[\lim_{k\to\infty}\frac{\ln\left(1+\frac{r}{k}\right)}{1/k}\right] \\
&= \exp\left[\lim_{k\to\infty}\frac{\left(-r/k^2\right)\left(1/\left(1+\frac{r}{k}\right)\right)}{-1/k^2}\right] \\
&= \exp\left[\lim_{k\to\infty}\frac{r}{1+\frac{r}{k}}\right] \\
&= \exp[r] = e^r.
\end{aligned}$$

Problem 3. Euler's Gamma Function $\Gamma(x)$ uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t, \quad x > 0.$$

That is, for each positive x, the number $\Gamma(x)$ is the integral of the function $f(t) := t^{x-1}e^{-t}$ with respect to t over the interval $t \in [0,\infty)$. This definition can be extended to negative noninteger values of x by using the formula $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$, which we will confirm for the nonnegative integers in this exercise.

(a) **[5 points]** Show that $\Gamma(1) = 1$.

Solution.

$$\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}\,\mathrm{d}t = \int_0^\infty e^{-t}\,\mathrm{d}t = \lim_{b\to\infty}\int_0^b e^{-t}\,\mathrm{d}t = \lim_{b\to\infty}\left[-e^{-t}\right]_0^b = \lim_{b\to\infty}\left[-e^{-b}+e^0\right] = 1.$$

(b) **[4 points]** Apply integration by parts to the integral for $\Gamma(x+1)$ to show that $\Gamma(x+1) = x\Gamma(x)$. This gives the sequence:

$$\begin{aligned}
\Gamma(2) &= 1\Gamma(1) = 1 \\
\Gamma(3) &= 2\Gamma(2) = 2 \\
\Gamma(4) &= 3\Gamma(3) = 6 \\
&\vdots \\
\Gamma(n+1) &= n\Gamma(n) = n!
\end{aligned}$$

Solution.

$$\Gamma(x+1) = \int_0^\infty t^{(x+1)-1} e^{-t}\ \mathrm{d}t = \int_0^\infty t^x e^{-t}\ \mathrm{d}t = \lim_{b\to\infty} \int_0^b t^x e^{-t}\ \mathrm{d}t$$

Let $u := t^x$, $\mathrm{d}u := xt^{x-1}\ \mathrm{d}t$, $v := -e^{-t}$, and $\mathrm{d}v := e^{-t}\ \mathrm{d}t$, so that by the integration by parts formula ($\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u$), we have

$$\begin{aligned}
\Gamma(x+1) &= \lim_{b\to\infty} \left[\left[-t^x e^{-t} \right]_{t=0}^{t=b} - \int_0^b (-e^{-t}) x t^{x-1}\ \mathrm{d}t \right] \\
&= \lim_{b\to\infty} \left[-b^x e^{-b} + x \int_0^b t^{x-1} e^{-t}\ \mathrm{d}t \right] \\
&= \lim_{b\to\infty} \left[-b^x e^{-b} \right] + \lim_{b\to\infty} \left[x \int_0^b t^{x-1} e^{-t}\ \mathrm{d}t \right].
\end{aligned}$$

Now, we observe that in the limit on the left, the e^{-b} factor grows more quickly than the b^x factor, so it will dominate and the limit should be zero; we could also reason that to the limit of the function $(-b^x/e^b)$, which has the form "$\frac{\infty}{\infty}$", l'Hôpital's rule can be successively applied (indeed, the numerators are all polynomial functions of b that are infinitely differentiable, and the denominators remain e^b which is also infinitely differentiable, and which never has derivative zero), and after x many iterations of l'Hôpital's rule, we end up with the limit $\lim_{b\to\infty} \left(1/e^b\right)$, which is zero. Therefore,

$$\begin{aligned}
\Gamma(x+1) &= 0 + \lim_{b\to\infty} \left[x \int_0^b t^{x-1} e^{-t}\ \mathrm{d}t \right] \\
&= x \int_0^\infty t^{x-1} e^{-t}\ \mathrm{d}t \\
&= x\Gamma(x).
\end{aligned}$$

(c) **[1 point]** Use the principle of mathematical induction[2] to show that the above sequence holds for every nonnegative integer n.

Solution. We have proven that $\Gamma(1) = 1 = 0!$, and that for all $n \in \mathbb{N}$, $\Gamma(n+1) = n\Gamma(n)$; that is,

$$\Gamma(n+1) = n\Gamma(n).$$

Therefore, if for some fixed $n \in \mathbb{N}$, we suppose that $\Gamma(n) = (n-1)!$, then we may conclude $\Gamma(n+1) = n(n+1)! = n!$. Therefore, since the base case $\Gamma(1) = 0! = 1$ holds, we may apply the induction hypothesis successively to determine that, by the principle of mathematical induction, we for all $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.

[2]Once you've done parts (a) and (b), this is a very easy step; if you have never used the principle of mathematical induction before, then please look at the first section (the first five lines) of this document: `http://www.cs.cornell.edu/courses/cs2800/2014fa/handouts/toronto_induction.pdf`. The document also contains some fun mathematics problems at the end, but these are not relevant for this course.

Problem 4. This problem is intended to show that $\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x$ is not necessarily equal to $\lim_{c\to\infty} \int_{-c}^{c} f(x)\ \mathrm{d}x$.

(a) [**4 points**] Show that $\displaystyle\int_0^{\infty} \frac{2x\ \mathrm{d}x}{x^2+1}$ diverges, and conclude that $\displaystyle\int_{-\infty}^{\infty} \frac{2x\ \mathrm{d}x}{x^2+1}$ diverges as well.

Solution.

$$\int_0^{\infty} \frac{2x\ \mathrm{d}x}{x^2+1} = \lim_{b\to\infty} \int_0^{b} \frac{2x\ \mathrm{d}x}{x^2+1},$$

and if we let $u := x^2+1$ and $\mathrm{d}u := 2x\ \mathrm{d}x$, we may perform u-substitution to obtain

$$\int_0^{\infty} \frac{2x\ \mathrm{d}x}{x^2+1} = \lim_{b\to\infty} \int_1^{b^2+1} \frac{\mathrm{d}u}{u} = \lim_{b\to\infty} [\ln|u|]_0^{b^2+1} = \lim_{b\to\infty} \ln\left|b^2+1\right| - \ln|1| = \lim_{b\to\infty} \ln(b^2+1) \to +\infty,$$

where the absolute value signs were removed in the next-to-last expression because the expression b^2+1 is always positive. We therefore conclude that both improper integrals in the problem statement diverge.

(b) [**3 points**] Show that $\displaystyle\lim_{c\to\infty} \int_{-c}^{c} \frac{2x\ \mathrm{d}x}{x^2+1} = 0.$

Solution. Notice that the integrand, $\dfrac{2x}{x^2+1}$, is an odd function of x, and that the (finite!) interval of integration, $(-c, c)$, is symmetric about $x = 0$. Therefore, since the integral of an odd function over an interval symmetric about 0 is 0, we may conclude that

$$\lim_{c\to\infty} \int_{-c}^{c} \frac{2x\ \mathrm{d}x}{x^2+1} = \lim_{c\to\infty} 0 = 0.$$

(c) [**3 points**] Using the definitions on Page 505, write a correct expression for $\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x$, where $f(x)$ is continuous on $(-\infty, \infty)$. Your expression should involve two different limits with two different limiting variables (do not just copy item 3 from the definition; use items 1 and 2 to expand it).

Solution. From Page 505, item 4 states that for a continuous function $f(x)$ and some arbitrary $c \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = \int_{-\infty}^{c} f(x)\ \mathrm{d}x + \int_{c}^{\infty} f(x)\ \mathrm{d}x.$$

We now simply plug in item 1 and item 2 to expand the first and second terms, respectively, of the sum on the right-hand side:

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = \int_{-\infty}^{c} f(x)\ \mathrm{d}x + \int_{c}^{\infty} f(x)\ \mathrm{d}x = \lim_{a\to-\infty} \int_{a}^{c} f(x)\ \mathrm{d}x + \lim_{b\to\infty} \int_{c}^{b} f(x)\ \mathrm{d}x.$$

Problem 5. [10 points] Use the definition of convergence on Page 574 of the text to prove that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$. Hint: $|\sin(x)| < 1$ for all x.

Solution. This problem asked us to use the definition of convergence, which is that a sequence $\{a_n\}$ converges to L if:

$$\boxed{\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that for } n \in \mathbb{N},\ n > N \implies |a_n - L| < \varepsilon.}$$

Now, before we begin our proof, we start computing; if we seek to prove that $\frac{\sin n}{n} \to 0$, then we should fix $\varepsilon > 0$ and see what needs to be true of a fixed value of n in order for $\left|\frac{\sin n}{n} - 0\right|$ to be less than some fixed positive ε. Well,

$$\left|\frac{\sin n}{n} - 0\right| = \left|\frac{\sin n}{n}\right| = \frac{|\sin n|}{n} \le \frac{1}{n},$$

where the $1/n$ is taken outside of the absolute value signs because $n \in \mathbb{N}$ implies that $1/n$ is always positive, and where we used $|\sin n| \le 1$ from the hint. Now, it is clear that in order to have $\left|\frac{\sin n}{n} - 0\right| < \varepsilon$, we must have $\frac{1}{n} < \varepsilon$; that is, $n > \frac{1}{\varepsilon}$. So we set $N := \text{ceil}(\frac{1}{\varepsilon})$, where the "ceil" function maps each number to the smallest integer that exceeds it. This is the end of our personal scratch work, and we may now start our proof:

Solution. Fix $\varepsilon > 0$, let $N := \text{ceil}(\frac{1}{\varepsilon})$, and suppose that for some $n \in \mathbb{N}$, $n > N$. Then

$$\left|\frac{\sin n}{n} - 0\right| = \left|\frac{\sin n}{n}\right| = \frac{|\sin n|}{n} \le \frac{1}{n} < \frac{1}{N} \le \varepsilon.$$

So $\left|\frac{\sin n}{n} - 0\right| < \varepsilon$, and we have proven that $\frac{\sin n}{n} \to 0$.

Problem 6. Newton's method, applied to a differentiable function $f(x)$, begins with a starting value x_0 and generates from it a sequence of numbers $\{x_n\}$ that, under favorable circumstances, converges to a zero of f. The recursion formula for the sequence is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(a) **[5 points]** Show that the recursion formula for $f(x) = x^2 - a$, for $a > 0$ constant, can be written as $x_{n+1} = \dfrac{x_n + \frac{a}{x_n}}{2}$.

Solution. Well, if $f(x) = x^2 - a$, then $f'(x) = 2x$, and so the recursion formula for our sequence becomes:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{2x_n^2 - (x_n - a)}{2x_n} = \frac{2x_n^2 - x_n^2 + a}{2x_n} = \frac{x_n^2 + a}{2x_n} = \frac{x_n + \frac{a}{x_n}}{2}.$$

(b) **[5 points]** Starting with $x_0 = 1$ and $a = 3$, calculate successive terms of the sequence until the display on your calculator or computer prompt begins to repeat. What number is being approximated? Please refer to the function $f(x)$ in your answer.

Solution. If we take $a = 3$, then our recursion formula becomes $x_{n+1} = \dfrac{x_n + \frac{3}{x_n}}{2}$. I chose to write a MATLAB program to iterate this sequence:

```
function Prob6(x_0)
% function Prob6(x_0)
%
% Performs Newton iteration for the function f(x)=x^2-3 using the starting
% value x_0. Stopping condition is when display repeats.

a=3; % the given a-value
N=@(x) 0.5*(x+a/x); % the computed recursion formula

x_curr=x_0; % the input x_0 value
x_new=N(x_curr); % we compute x_1
its=1; % we create an iterate counter and set it to 1
fprintf('x_0 = %17.15g\nx_1 = %17.15g\n',x_curr,x_new); % prints x_0 and x_1
while abs(x_curr-x_new)>1e-15 % format(long) in MATLAB displays 15 digits,
                              % so want difference less than 1e-15
  its=its+1; % increment the iterate counter
  x_curr=x_new; % the 'new' x-value from the last iteration becomes current x-value
  x_new=N(x_curr); % we compute the new x-value for this iteration
  fprintf('x_%g = %17.15g\n',its,x_new); % prints x_n
end
```

And this code gave the following output when run:

```
>> format long
>> format compact
>> Prob6(1)
x_0 =                 1
x_1 =                 2
x_2 =              1.75
x_3 =  1.73214285714286
x_4 =  1.73205081001473
x_5 =  1.73205080756888
```

```
x_6 =  1.73205080756888
```

So, it looks like Newton's method is converging (and it's converging fast! It took only six iterations to get 15 accurate decimal places!). We were told in the problem statement that Newton's method, under favorable circumstances (which it looks like we're meeting), converges to a zero of $f(x)$, which for us was $f(x) = x^2 - a = x^2 - 3 = (x+3)(x-3)$, which has zeros $\pm\sqrt{3}$. Obviously, from our starting point $x_0 = 1$, it looks like Newton's method is converging to the positive root, so we can say that the recursively defined sequence $x_0 = 1$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{n_n}\right)$ converges to $\sqrt{3}$.
To see convergence to the negative root of the function, we just change the starting point:

```
>> Prob6(-1)
x_0 =                 -1
x_1 =                 -2
x_2 =              -1.75
x_3 = -1.73214285714286
x_4 = -1.73205081001473
x_5 = -1.73205080756888
x_6 = -1.73205080756888
```