Calculus III
E2 Term, Sections E201 and E296
Instructor: E.M. Kiley
Due Wednesday, August 05, 2015, 11:59 p.m. EDT

Week 4: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it.

Reading

Please read Sections 11.1, and 11.2 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 11, "Questions to Guide Your Review", p. 647, Problems 1, 2, 4–6, 8

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 11.1, Problems 1–37 odd
- Section 11.2, Problems 1–35 odd

[1] See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

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Name: ____________________

WPI Username: ____________________

Week 4: Homework Problems

Due date: Wednesday, August 5, 2015, 11:59 p.m. EDT. Please upload a single `.pdf` document to myWPI (`my.wpi.edu`).

Rules for Calculus Assignments:

I) Each student must compose his or her assignments independently. However, brainstorming may be done in groups.

II) Please typeset your solutions using LaTeX, or handwrite them neatly and legibly.

III) **Show your work**. Explain your answers using **full English sentences**.

IV) **No late assignments will be accepted for credit.**

Problem 1. Find a parametrization for the following curves:

(a) [**2 points**] The line segment with endpoints $(-1, -3)$ and $(3, 2)$.

Solution. Of course, there are always infinitely many ways to parametrize a curve. The sample solutions to these problems demonstrate just one way.
I chose to parametrize this curve by setting $x = a + bt$, and $y = c + dt$, then stipulating that at $t = 0$, the curve would begin at the point $(-1, -3)$ and when $t = 1$, the curve would end at the point $(3, 2)$. When plugging in $t = 0$, and setting $(x, y) = (-1, -3)$, we obtain the two equations $a = -1$ and $c = -3$, so $x = bt - 1$ and $y = dt - 3$. When substituting in $t = 1$ and setting $(x, y) = (3, 2)$, we obtain the equations $b = 3 - (-1) = 4$ and $d = 2 - (-3) = 5$. Therefore, we obtain the parametrization

$$\begin{cases} x = 4t - 1 \\ y = 5t - 3 \\ t \in [0, 1] \end{cases}$$

(b) [**2 points**] The ray (half line) with initial point $(2, 3)$ that also passes through the point $(-1, 1)$.

Solution. I chose to parametrize this curve by again setting $x = a+bt$, and $y = c+dt$, then stipulating that at $t = 0$, the curve would begin at the point $(2, 3)$ and when $t = 1$, the curve would pass through the point $(-1, 1)$. When plugging in $t = 0$, and setting $(x, y) = (2, 3)$, we obtain the two equations $a = 2$ and $c = 3$, so $x = 2 + bt$ and $y = 3 + dt$. When substituting in $t = 1$ and setting $(x, y) = (-1, 1)$, we obtain the equations $b = -1 - (2) = -3$ and $d = 1 - (3) = -2$. Therefore, we obtain the parametrization

$$\begin{cases} x = 2t - 3 \\ y = 3t - 2 \\ t \in [0, +\infty) \end{cases}$$

(c) [**3 points**] The lower half of the parabola $x - 2 = 2y^2$.

Solution. I chose to parametrize this curve by setting $y = t$ and $x = 2t^2 + 2$. The vertex of the parabola is at $(-2, 0)$, and in order to obtain the lower half, we choose t-values less than or equal to the y-coordinate (that is, less than or equal to zero). Therefore, we obtain the parametrization

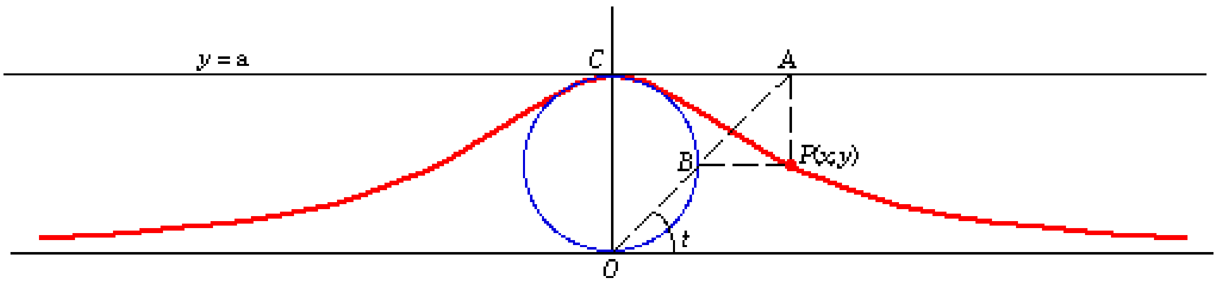
$$\begin{cases} x = 2t^2 + 2 \\ y = t \\ t \in (-\infty, 0] \end{cases}$$

(d) **[3 points]** The left-hand half of the parabola $y = x^2 + 2x$.

Solution. I chose to parametrize this curve by setting $x = t$, and $y = t^2 + 2t$ (the natural parametrization). The vertex of the parabola is at $(x, y) = (-1, -1)$, and in order to obtain the left-hand half, we choose t-values less than or equal to the x-coordinate (that is, less than or equal to -1). Therefore, we obtain the parametrization

$$\begin{cases} x = t \\ y = 2t^2 + 2t \\ t \in (-\infty, -1] \end{cases}$$

Problem 2. **[10 points]** The bell-shaped witches of Maria Agnesi[2] form a set of curves; we will construct one of them in the following way. Starting with a circle of radius 1 and centered at the point $(0,1)$, choose a point A on the line $y=2$ and connect it to the origin with a line segment. Call the point where the segment crosses the circle B, and let P be the point where the vertical line through A crosses the horizontal line through B. The witch[3] is the curve traced by P as A moves along the line $y=2$:



Find parametric equations and a parameter interval for the witch by expressing the coordinates of P in terms of t, the radian measure of the angle that the segment OA makes with the positive x-axis. It may help if you use the following inequalities (which you do not have to prove):

$$x = AC, \qquad y = 2 - AB\sin t, \qquad AB \cdot OA = (AC)^2.$$

Remember that on this and all homework problems, **you must show your work**. If you write down the parametric equations with no explanation, you will get zero credit.

Solution. The entire problem here was to find x and y in terms of t only; we use trigonometry of right triangles to accomplish this. First, observe that $OC = 2$ is a leg of the triangle $\triangle OAC$, and that angle $\angle OAC$ of this triangle has measure t (since $\angle OAC$ is an alternate interior angle to the one we've labelled t in the diagram). Therefore, $\tan(t) = \frac{2}{AC}$, and so $AC = \frac{2}{\tan(t)}$.
Also observe that OA is the hypotenuse of $\triangle OAC$, and as such, $\sin(t) = \frac{2}{OA}$, which implies $OA = \frac{2}{\sin(t)}$.
We were given that $AB \cdot OA = (AC)^2$, and so

$$AB = \frac{(AC)^2}{OA} = \frac{4/\tan^2(t)}{2/\sin(t)} = \frac{2\sin(t)}{\tan^2(t)} = \frac{2\sin(t)\cos^2(t)}{\sin^2(t)} = \frac{2\cos^2(t)}{\sin(t)}.$$

Therefore, using the facts we were given, we obtain

$$x = AC = \frac{2}{\tan(t)} = 2\cot(t) = \frac{2\cos(t)}{\sin(t)},$$

and

$$y = 2 - AB\sin t = 2 - \frac{2\cos^2(t)}{\sin(t)}\sin(t) = 2 - \cos^2(t).$$

Now, observe that the range $t \in [0,\pi]$ sweeps out the entire range of values we are interested in, and so the parametrization becomes, finally,

$$\begin{cases} x = 2\cot t \\ y = 2 - \cos^2 t \\ t \in [0,\pi] \end{cases}.$$

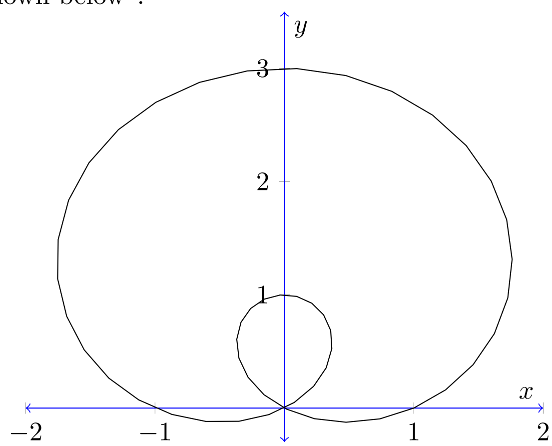
[2] Agnesi was an interesting figure in mathematics; the curious can read a little about her life here: `https://en.wikipedia.org/wiki/Maria_Gaetana_Agnesi` .

[3] For a `.gif` of this construction process, see `http://mathworld.wolfram.com/images/gifs/WitchOfAgnesi.gif` .

Problem 3. The curve with parametric equations

$$x = (1 + 2\sin\theta)\cos\theta, \qquad y = (1 + 2\sin\theta)\sin\theta$$

is called a *limaçon* and is shown below[4].



Find the points (x, y) and the slopes of the tangent lines at these points for the following values of the parameter θ:

(a) **[3 points]** $\theta = 0$.

Solution. The point (x, y) is given by the parametrization:

$$x = (1 + 2\sin(0))\cos(0) = (1 + 2\cdot 0)1 = 1; \quad y = (1 + 2\sin(0))\sin(0) = (1 + 2\cdot 0)0 = 0,$$

and so is $(1, 0)$, whereas the slope of the tangent line is given by the formula

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[(1 + 2\sin\theta)\sin\theta\right]}{\frac{\mathrm{d}}{\mathrm{d}t}\left[(1 + 2\cos\theta)\sin\theta\right]} = \frac{(1 + 2\sin\theta)\cos\theta + 2\cos^2\theta}{(1 + 2\sin\theta)\cos\theta - 2\sin^2\theta} = \frac{\cos\theta + 2\cos^2\theta + 2\sin\theta\cos\theta}{\cos\theta - 2\sin^2\theta + 2\sin\theta\cos\theta},$$

and plugging in $\theta = 0$ yields

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=0} = \frac{\cos(0) + 2\cos^2(0) + 2\sin(0)\cos(0)}{\cos(0) - 2\sin^2(0) + 2\sin(0)\cos(0)} = \frac{1 + 2\cdot 1 + 2\cdot 0\cdot 1}{1 - 2(0)^2 + 2\cdot 0\cdot 0} = \frac{3}{1} = 3.$$

Therefore, at $\theta = 0$, the point is $(x, y) = (1, 0)$, and the slope of the tangent line is 3.

(b) **[3 points]** $\theta = \pi/2$.

Solution. The point (x, y) is given by the parametrization:

$$x = (1 + 2\sin(\pi/2))\cos(\pi/2) = (1 + 2\cdot 1)0 = 0; \quad y = (1 + 2\sin(\pi/2))\sin(\pi/2) = (1 + 2\cdot 1)1 = 3,$$

and so is $(0, 3)$, whereas the slope of the tangent line is given by the same formula as before, and substituting $\theta = \pi/2$ yields

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\pi/2} = \frac{\cos(\pi/2) + 2\cos^2(\pi/2) + 2\sin(\pi/2)\cos(\pi/2)}{\cos(\pi/2) - 2\sin^2(\pi/2) + 2\sin(\pi/2)\cos(\pi/2)} = \frac{0 + 2\cdot 0^2 + 2\cdot 1\cdot 0}{0 - 2\cdot 1^2 + 2\cdot 1\cdot 0} = \frac{0}{-2} = 0.$$

Therefore, at $\theta = \pi/2$, the point is $(x, y) = (0, 3)$, and the slope of the tangent line is 0.

[4] A `.gif` of the process of drawing a limaçon is here: `https://en.wikipedia.org/wiki/File:PedalCurve2.gif`.

(c) **[4 points]** $\theta = 4\pi/3$. For this θ value, please also write an equation for the line tangent to the limaçon at the point (x, y). (The equation of the tangent line can be in Cartesian coordinates—no need to parametrize.)

Solution. The point (x, y) is given by the parametrization:

$$x = \left(1 + 2\sin\left(\frac{4\pi}{3}\right)\right)\cos\left(\frac{4\pi}{3}\right) = \left(1 + 2\cdot\frac{-\sqrt{3}}{2}\right)\frac{-1}{2} = \frac{\sqrt{3}-1}{2}$$

$$y = \left(1 + 2\sin\left(\frac{4\pi}{3}\right)\right)\sin\left(\frac{4\pi}{3}\right) = \left(1 + 2\cdot\frac{-\sqrt{3}}{2}\right)\frac{-\sqrt{3}}{2} = \frac{3-\sqrt{3}}{2},$$

and so is $\left(\frac{\sqrt{3}-1}{2}, \frac{3-\sqrt{3}}{2}\right)$, whereas the slope of the tangent line is given by the same formula as before, and substituting $\theta = 4\pi/3$ yields

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{4\pi}{3}} = \frac{\cos\left(\frac{4\pi}{3}\right) + 2\cos^2\left(\frac{4\pi}{3}\right) + 2\sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{4\pi}{3}\right)}{\cos\left(\frac{4\pi}{3}\right) - 2\sin^2\left(\frac{4\pi}{3}\right) + 2\sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{4\pi}{3}\right)} = \frac{2\sqrt{3}-1}{\sqrt{3}+\frac{5}{2}}.$$

Therefore, at $\theta = 4\pi/3$, the point is $(x, y) = \left(\frac{\sqrt{3}-1}{2}, \frac{3-\sqrt{3}}{2}\right)$, and the slope of the tangent line is $\frac{2\sqrt{3}-1}{\sqrt{3}+\frac{5}{2}}$. The equation of the tangent line is therefore:

$$y - \frac{3-\sqrt{3}}{2} = \frac{2\sqrt{3}-1}{\sqrt{3}+\frac{5}{2}}\left(x - \frac{\sqrt{3}-1}{2}\right).$$

Problem 4. To illustrate the fact that the numbers we get for curve length do not depend on the way we parametrize our curves (except for the restrictions preventing doubling back mentioned in the definition of the arc length), calculate the length of the semicircle $y = \sqrt{1-x^2}$ with these two different parametrizations:

(a) **[5 points]** $x = \cos(2t)$, $y = \sin(2t)$, $t \in [0, \pi/2]$.

Solution. The curve is smooth and is traced exactly once as t ranges from 0 to $\pi/2$, and so the arc length is given by the formula

$$L = \int_0^{\pi/2}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^{\pi/2}\sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2}\,\mathrm{d}t$$

$$= \int_0^{\pi/2}\sqrt{4\left(\cos^2(2t) + \sin^2(2t)\right)}\,\mathrm{d}t = \int_0^{\pi/2} 2\,\mathrm{d}t = 2\left(\frac{\pi}{2} - 0\right) = \pi.$$

(b) **[5 points]** $x = \sin(\pi t)$, $y = \cos(\pi t)$, $t \in [-1/2, 1/2]$.

Solution. The curve is smooth and is traced exactly once as t ranges from $-1/2$ to $1/2$, and so the arc length is given by the formula

$$L = \int_{-1/2}^{1/2}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_{-1/2}^{1/2}\sqrt{(\pi\cos(\pi t))^2 + (\pi\sin(\pi t))^2}\,\mathrm{d}t$$

$$= \int_{-1/2}^{1/2}\sqrt{\pi^2\left(\cos^2(\pi t) + \sin^2(\pi t)\right)}\,\mathrm{d}t = \int_{-1/2}^{\pi/2}\pi\,\mathrm{d}t = \pi\left(\frac{1}{2} - \frac{-1}{2}\right) = \pi.$$

Problem 5. **[10 points]** The line segment joining the origin to the point (h, r) is revolved about the x-axis to generate a cone of height h and base radius r. Find the cone's surface area with the parametric equations $x = ht$, $y = rt$, $0 \le t \le 1$. Check your result against the standard formula: $A = \pi r\cdot$(slant height).

Solution. The line segment is a smooth curve, and is traced out exactly once as t varies from 0 to 1, so we may apply for formula for revolving about the x-axis:

$$A = \int_0^1 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^1 2\pi(rt)\sqrt{h^2+r^2}\,\mathrm{d}t = 2\pi r\sqrt{h^2+r^2}\left[\frac{1}{2}t^2\right]\Bigg|_0^1 = \pi r\sqrt{h^2+r^2}.$$

Now, notice that the cross-section of a cone is an equilateral triangle whose height is h and whose base is $2r$; we draw a line along the axis and split this into two right triangles, each with leg h and base r, and using the Pythagorean theorem on either of these right triangles, we find that the hypotenuse—that is, the slant height of the cone—is given by $\sqrt{r^2+h^2}$. So the standard formula $A = \pi r \cdot$(slant height) is equivalent to $A = \pi r\sqrt{h^2+r^2}$, which is just what we obtained using the integration formula for the surface area of a solid of revolution.

Problem 6. [10 points] Find the area under one arch of the cycloid

$$x = a(t - \sin t), \qquad y = a(1 - \cos t).$$

Solution. The curve is smooth, and we notice that exactly one arch of the cycloid is traced out as t varies from 0 to 2π. Therefore, the area between the curve and the x-axis is

$$A = \int_0^{2\pi} y \,\mathrm{d}x = \int_0^{2\pi} [a(1-\cos t)]\,[a(1-\cos t)\,\mathrm{d}t] = a^2 \int_0^{2\pi} (1-\cos t)^2 \,\mathrm{d}t$$
$$= a^2 \int_0^{2\pi} 1 - 2\cos(t) + \frac{1}{2}[1+\cos(2t)]\,\mathrm{d}t = a^2 \int_0^{2\pi} \frac{3}{2} - 2\cos(t) + \frac{1}{2}\cos(2t)\,\mathrm{d}t = \left[\frac{3}{2}t - 2\sin(t) + \frac{1}{4}\sin(2t)\right]\Big|_0^{2\pi} = a^2 \left[3\pi -\right.$$