Calculus III
E2 Term, Sections E201 and E296
Instructor: E.M. Kiley
Due Wednesday, August 19, 2015, 11:59 p.m. EDT

Week 6: Reading, Practice Problems, and Homework Exercises

Reminder

Your submitted homework solutions should show not only your answers, but should show a clearly reasoned logical argument, written using **complete English sentences**, leading to that solution. Each mathematical symbol that you will encounter stands for one or more English words[1], and if you elect to use any symbols, you should do so *only* in full sentences where you intend to abbreviate words.

If the work that you submit is incomplete or illegible, you will not receive credit for it.

Reading

Please read Sections 12.5 and 13.1 in time for Tuesday's lecture, and 13.2, 13.3, and 13.4 in time for Thursday's lecture. (In-class students, you can always re-watch the lectures online after you finish your reading, if it would benefit you.) I will not necessarily cover all of this material in class, but you will be responsible for it. Any questions about any of the material can be addressed in class or office hours, or to me via e-mail (`emkiley@wpi.edu`).

Questions to Guide Your Review

Note: Do not hand these in!

Please find at the end of each chapter, before the chapter problems are given, the "Questions to Guide Your Review" section. You should read through these items to check your understanding of the chapter, but you are not required to hand in your answers. If you have questions about these, you will usually be able to find your answer by re-reading the section, by consulting the hints in the back of the book, or, if you are really stuck, by consulting me. These are meant to be conceptually important questions for you to check how well you have understood the material in each section, and if you expect to do well on the midterm and final exams, I suggest studying these in particular.

The relevant questions for this week's material are:

- Chapter 12, "Questions to Guide Your Review", p. 745, Problems 9, 11, 12
- Chapter 13, "Questions to Guide Your Review", p. 788, Problems 1–11

Practice Problems

Note: Do not hand these in!

Here are some practice problems to work on at home. It is extremely important that you are proficient at exercises such as these; without the basic skills, you will find it difficult to complete your exams in the allotted time.

You will find the answers to the odd-numbered problems in the back of the book. This is useful if you want to check your work, but please remember that the *logical argument*, not the final answer, is the most important part of solving a problem for credit in this class. You should therefore understand *how to solve* each of these problems. In particular, you should *not* be satisfied with merely looking up the solution in the back of the book.

Please discuss any questions with me in class, during my office hours, or send me an e-mail.

- Section 12.5, Problems 1–17 odd; 21–59 odd
- Section 13.1, Problems 1–23 odd
- Section 13.2, Problems 1–37 odd
- Section 13.3, Problems 1–15 odd
- Section 13.4, Problems 1–17 odd

[1] See a list of mathematical symbols and their meanings here: `http://en.wikipedia.org/wiki/List_of_mathematical_symbols`

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Name: ______________________

WPI Username: ________________

Week 6: Homework Problems

Due date: Wednesday, August 19, 2015, 11:59 p.m. EDT. Please upload a single `.pdf` document to myWPI (`my.wpi.edu`).

Rules for Calculus Assignments:

- **I)** Each student must compose his or her assignments independently. However, brainstorming may be done in groups.
- **II)** Please typeset your solutions using LaTeX, or handwrite them neatly and legibly.
- **III)** **Show your work**. Explain your answers using **full English sentences**.
- **IV)** **No late assignments will be accepted for credit.**

Problem 1. **[10 points]** Prove that four points A, B, C, and D are coplanar (i.e., they lie in a single common plane) if and only if $\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{BC}\right) = 0$. [Hint: Remember that there are two things to prove for an "if and only if" statement: the forward direction of the implication, and also the backward direction.]

Solution. We first prove the forward direction; assume that A, B, C, and D are coplanar. Then $\overrightarrow{AD}$ lies in the same plane as $\overrightarrow{AB}$ and $\overrightarrow{BC}$, so the cross product $\left(\overrightarrow{AB} \times \overrightarrow{BC}\right)$ is normal to $\overrightarrow{AD}$; that is, $\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{BC}\right) = 0$.

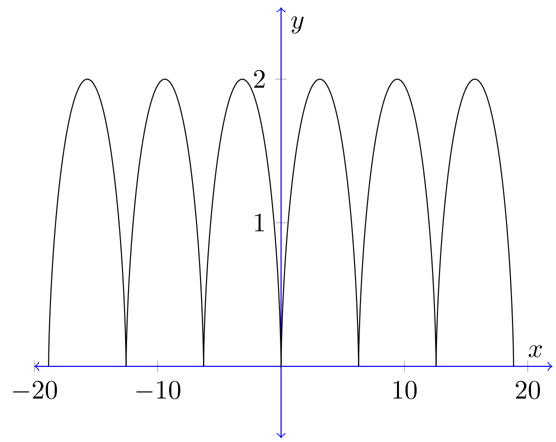
To prove the backward direction, assume that for some points in the plane, $\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{BC}\right) = 0$. We know that any three points—say, the points A, B, and C—in space define a plane, so we seek to show that D also happens to be in the plane defined by A, B, and C. Recall that the cross product of any two vectors in a plane is orthogonal to the plane that contains them—that is, $\left(\overrightarrow{AB} \times \overrightarrow{BC}\right)$ is orthogonal to the plane defined by A, B, and C, and every other vector that lies within this plane will also be orthogonal to that dot product. But we knew that $\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{BC}\right) = 0$, so $\overrightarrow{AD}$ lies in the plane that contains A, B, and C, and as a consequence, the point D is coplanar with A, B, and C.

Problem 2. Suppose that a particle moves in the xy-plane in such a way that its position at time t is given by

$$\vec{r}(t) = (t - \sin t)\,\hat{\imath} + (1 - \cos t)\,\hat{\jmath}.$$

(a) [**3 points**] Graph $\vec{r}(t)$. [Hint: This curve is called a cycloid, and you should be able to graph it without help, but you may use a graphing calculator or computational aid if you wish.]

Solution.



(b) [**7 points**] Find the maximum and minimum values of $|\vec{v}|$ and $|\vec{a}|$, where $\vec{v}$ and $\vec{a}$ represent the velocity and the acceleration, respectively, of the particle. [Hint: Find the extrema of $|\vec{v}|^2$ and $|\vec{a}|^2$ first, and then take square roots later.]

Solution. The velocity of the particle is given by $\vec{v}(t) = \vec{r}'(t) = (1 - \cos t)\,\hat{\imath} + (\sin t)\hat{\jmath}$, and the acceleration by $\vec{a}(t) = \vec{v}'(t) = (\sin t)\,\hat{\imath} + (\cos t)\hat{\jmath}$. Therefore,

$$|\vec{v}| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2}\sqrt{1 - \cos t},$$

and $\sqrt{2}\sqrt{1 - \cos t}$ has extrema exactly where $1 - \cos t$ does; namely, the maximum value occurs when $1 - \cos t$ is maximum—that is, when $t = (2n + 1)\pi$ $(n \in \mathbb{N})$, and $1 - \cos t = 2$, so that the maximum speed is $|\vec{v}| = \sqrt{2}\sqrt{2} = 2$, and the minimum value occurs when $1 - \cos t$ is minimum—that is, when $t = 2n\pi$ $(n \in \mathbb{N})$, and $1 - \cos t = 0$, so that the minimum speed is $|\vec{v}| = \sqrt{2}\sqrt{0} = 0$.

The magnitude of the acceleration is constant:

$$|\vec{a}| = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1,$$

and so the maximum and minimum values of acceleration are both 1.

Problem 3. **[7 points]** An electron in a cathode ray tube is beamed horizontally at a speed of 5×10^6 meters per second toward the inside face of a television 40 cm away. About how far will the electron drop before it hits?

Solution. We assume the equations of ideal projectile motion; that is, taking $v_0 = 5 \times 10^6$ m/s and $\alpha = 0$, the position should be given by

$$\vec{r}(t) = (v_0 \cos\alpha)t\hat{\imath} + (v_0 t \sin\alpha - \frac{1}{2}gt^2)\hat{\jmath} = 5t \times 10^6 \hat{\imath} - \frac{1}{2}gt^2\hat{\jmath}.$$

The electron will hit the face of the television when the x-component is equal to 40 cm, i.e., when

$$5t \times 10^6 \text{ (m/s)} = 0.4 \text{ m} \iff t = \frac{4}{5} \times 10^{-7} \text{ s} = 8 \times 10^{-8} \text{ s},$$

and in that time, the y-component will be equal to

$$\frac{1}{2}g(8 \times 10^{-8})^2 = 32(9.8) \times 10^{-16} \text{ m} \approx 3.136 \times 10^{-14} \text{ m},$$

where we took $g \approx 9.8$ m/s^2.

Problem 4. A volleyball is hit when it is 4 feet above the ground and 12 feet from a 6-foot-high net. It leaves the point of impact with an initial velocity of 35 feet per second, at an angle of 27° from the horizontal, and slips by the opposing team untouched.

(a) **[2 points]** Find a vector equation for the path of the volleyball.

Solution. We use the initial point $(x_0, y_0) = (-12, 4)$, the initial velocity $v_0 = 35$ f/s, the angle $\alpha = 27°$, and the gravitational constant $g \approx 32.17$ f/s^2 in the standard formula from Equation 7 on page 763 of the text:

$$\begin{aligned}\vec{r}(t) &= [x_0 + (v_0 \cos\alpha)t]\,\hat{\imath} + \left[y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2\right]\hat{\jmath} \\ &= [(35\cos 27)t - 12]\,\hat{\imath} + \left[(35\sin 27)t - 16.85t^2 + 4\right]\hat{\jmath}.\end{aligned}$$

(b) **[3 points]** How high does the volleyball go, and when does it reach maximum height?

Solution. We maximize $y(t)$ by observing that it is a downward-pointing parabola whose vertex occurs at the only critical point of the function, computed by setting $y'(t) = 0$:

$$y'(t_{max}) = (35\sin 27) - gt_{max} = 0 \iff t_{max} = \frac{35\sin 27}{g} \approx 0.49 \text{ seconds}.$$

This is the time when the height is maximum, and the maximum height itself is

$$y(t_{max}) = y(0.49) = (35\sin 27)0.49 - 16.85(0.49)^2 + 4 \approx 7.74 \text{ feet}.$$

(c) **[3 points]** Find its range and flight time.

Solution. The flight time is the time t_f until $y(t_f) = 0$; that is,

$$y(t_f) = (35\sin 27)t_f - 16.85t_f^2 + 4 = 0 \iff t_f = \frac{-(35\sin 27) \pm \sqrt{(35\sin 27)^2 + 16g}}{-2g} \approx -0.2065 \text{ or } 1.15 \text{ seconds}.$$

We use the positive value among these, so that the flight time is 1.15 seconds. The range is the distance the ball travels, that is, $x(t_f)$:

$$x(t_f) = (35\cos 27)t_f - 12 \approx (35\cos 27)(1.15) \approx 23.85 \text{ feet beyond the net, or } 35.85 \text{ feet from where it was hit}$$

(d) **[3 points]** When is the volleyball 7 feet above the ground? How far (ground distance) is the volleyball from where it will land?

Solution. The time when the ball is 7 feet above the ground is the time t_7 until $y(t_7) = 7$; that is,

$$y(t_7) = (35\sin 27)t_7 - 16.85t_7^2 + 4 = 7 \iff t_7 = \frac{-(35\sin 27) \pm \sqrt{(35\sin 27)^2 - 12g}}{-2g} \approx 0.2611 \text{ or } 0.6819 \text{ seconds.}$$

Naturally, since the ball travels in a downward-opening parabola shape, it will be at 7 feet twice during its motion. The range is the distance the ball travels until that time that is, $x(t_7)$:

$$x(t_7) = (35\cos 27)t_7 - 12 \approx (35\cos 27)(0.2611) \approx -3.858 \text{ ft, so } 3.858 \text{ ft before the net}$$

and

$$x(t_7) = (35\cos 27)t_7 - 12 \approx (35\cos 27)(0.6819) \approx 9.266 \text{ feet beyond the net}$$

(e) **[2 points]** Suppose the net is raised to 8 feet. Does this change things? Explain your answer.

Solution. Yes, this changes things, because in the initial scenario, the ball clears the net; if the net were raised to 8 feet, then the ball would not clear the net.

Problem 5. To illustrate that the length of a smooth space curve does not depend on the parametrization you use to compute it, calculate the length of one turn of the same helix with the three different parametrizations:

(a) **[4 points]** $\vec{r}(t) = (\cos 4t)\hat{\imath} + (\sin 4t)\hat{\jmath} + 4t\hat{k}, \quad 0 \le t \le \pi/2.$

Solution. The helix is smooth and the turn is traced exactly once as t varies between 0 and $\pi/2$, and so the length is given by

$$\begin{aligned} L &= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_0^{\pi/2} \sqrt{(-4\sin(4t))^2 + (4\cos(4t))^2 + (4)^2}\, dt \\ &= \int_0^{\pi/2} \sqrt{16(\sin^2(4t) + \cos^2(4t) + 1)}\, dt = \int_0^{\pi/2} 4\sqrt{2}\, dt = 4\sqrt{2}\left(\frac{\pi}{2} - 0\right) = 2\pi\sqrt{2}. \end{aligned}$$

(b) **[3 points]** $\vec{r}(t) = (\cos \frac{t}{2})\hat{\imath} + (\sin \frac{t}{2})\hat{\jmath} + \frac{t}{2}\hat{k}, \quad 0 \le t \le 4\pi.$

Solution. The helix is smooth and the turn is traced exactly once as t varies between 0 and 4π, and so the length is given by

$$\begin{aligned} L &= \int_0^{4\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_0^{4\pi} \sqrt{\left(-\frac{1}{2}\sin(\frac{t}{2})\right)^2 + \left(\frac{1}{2}\cos(\frac{t}{2})\right)^2 + \left(\frac{1}{2}\right)^2}\, dt \\ &= \int_0^{4\pi} \sqrt{\frac{1}{2}(\sin^2(4t) + \cos^2(4t) + 1)}\, dt = \int_0^{4\pi} \frac{\sqrt{2}}{2}\, dt = \frac{\sqrt{2}}{2}(4\pi - 0) = 2\pi\sqrt{2}. \end{aligned}$$

(c) **[3 points]** $\vec{r}(t) = (\cos t)\hat{\imath} - (\sin t)\hat{\jmath} - t\hat{k}, \quad -2\pi \le t \le 0.$

Solution. The helix is smooth and the turn is traced exactly once as t varies between -2π and 0, and so the length is given by

$$\begin{aligned} L &= \int_{-2\pi}^{0} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_{-2\pi}^{0} \sqrt{(-\sin(t))^2 + (-\cos(t))^2 + (-1)^2}\, dt \\ &= \int_{-2\pi}^{0} \sqrt{(\sin^2(t) + \cos^2(t) + 1)}\, dt = \int_{-2\pi}^{0} \sqrt{2}\, dt = \sqrt{2}\,(0 - (-2\pi)) = 2\pi\sqrt{2}. \end{aligned}$$

Problem 6. [10 points] Show that the parabola $y = ax^2$, where $a \neq 0$, has its largest curvature at its vertex and has no minimum curvature. [Note: Since the curvature of a curve remains the same if the curve is translated or rotated, this result is true for any parabola.]

Solution. We use the natural parametrization $x = t$, $y = at^2$, $t \in \mathbb{R}$, so that the position vector is

$$\vec{r}(t) = t\hat{\imath} + at^2\hat{\jmath}.$$

This implies that the velocity vector is

$$\vec{v}(t) = 1\hat{\imath} + 2at\hat{\jmath},$$

and the magnitude of the velocity vector is

$$|\vec{v}| = \sqrt{1 + 4a^2t^2}.$$

Therefore, the unit tangent vector is

$$\hat{T} = \frac{1}{\sqrt{1+4a^2t^2}}\hat{\imath} + \frac{2at}{\sqrt{1+4a^2t^2}}\hat{\jmath},$$

and the time derivative of the unit tangent vector is

$$\frac{\mathrm{d}\hat{T}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\sqrt{1+4a^2t^2}}\right]\hat{\imath} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{2at}{\sqrt{1+4a^2t^2}}\right]\hat{\jmath} = \left[\frac{-4a^2t}{(1+4a^2t^2)^{3/2}}\right]\hat{\imath} + \left[\frac{2a}{\sqrt{4a^2t^2}} - \frac{8a^3t^2}{(1+4a^2t^2)^{3/2}}\right]\hat{\jmath}.$$

The magnitude of the time derivative of the unit tangent vector is therefore

$$\begin{aligned}
\left|\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}\right| &= \sqrt{\left[\frac{-4a^2t}{(1+4a^2t^2)^{3/2}}\right]^2 + \left[\frac{2a}{\sqrt{1+4a^2t^2}} - \frac{8a^3t^2}{(1+4a^2t^2)^{3/2}}\right]^2} \\
&= \sqrt{\frac{16a^4t^2}{(1+4a^2t^2)^3} + \frac{4a^2}{1+4a^2t^2} - \frac{32a^4t^2}{(1+4a^2t^2)^2} + \frac{64a^6t^4}{(1+4a^2t^2)^3}} \\
&= a\sqrt{\frac{16a^2t^2}{(1+4a^2t^2)^3} + \frac{4}{1+4a^2t^2} - \frac{32a^2t^2}{(1+4a^2t^2)^2} + \frac{64a^4t^4}{(1+4a^2t^2)^3}} \\
&= a\sqrt{\frac{16a^2t^2 + 64a^4t^4 + 4(1+4a^2t^2)^2 - 32a^2t^2(1+4a^2t^2)}{(1+4a^2t^2)^3}} \\
&= \frac{a}{1+4a^2t^2}\sqrt{\frac{16a^2t^2 + 64a^4t^4 + 4 + 32a^2t^2 + 64a^4t^4 - 32a^2t^2 - 128a^4t^4}{1+4a^2t^2}} \\
&= \frac{a}{1+4a^2t^2}\sqrt{\frac{4(1+4a^2t^2)}{1+4a^2t^2}} \\
&= \frac{2a}{1+4a^2t^2}.
\end{aligned}$$

The curvature is thus

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}\right| = \frac{1}{\sqrt{1+4a^2t^2}} \cdot \frac{2a}{1+4a^2t^2} = \frac{2a}{(1+4a^2t^2)^{3/2}}.$$

Notice that the curvature will be maximum when the denominator is minimum; that is, when $t = 0$. Also, notice that as t increases, the denominator grows without bound, and so the curvature continues to approach zero. It therefore has no minimum value.