Lecture 7: Lesson and Activity Packet

MATH 232: Introduction to Statistics

October 3, 2016

Homework and Announcements

- Homework 6 due at 11:59 p.m. tonight on **Canvas**
  - Can re-submit until deadline!
- Quiz 4 on Wednesday
  - Generalized rule of counting
  - Permutations
  - Combinations ("n-choose-k")
  - Basic notions of probability (what is a sample space, what are events, what are outcomes?)
- Exam 1 on Friday
  - Review session on Wednesday after quiz
  - Topics are everything including today's material

Questions?

If not, then today, we continue **Chapter 4: Probability**.

Recall the basic concepts of probability:

Definition 1 (***Sample space***)
The sample space of an experiment, denoted S, is the set of all possible outcomes of the experiment.

Definition 2 (*Event*)
An event is any subset of the sample space, and is usually denoted E. A simple event is a set containing only one outcome, and a compound event is an event containing more than one outcome.

We now define the concept of probability itself:

Definition 3 (***Probability***)
Probability is a measure of the likelihood of the occurrence of some event. The probability of an event E is represented by the symbol $P(E)$, which is read, "P of E" or "the probability of the event E". For all events E, $P(E)$ is a real number between 0 and 1.

Example 1
If our experiment consists of tossing a single fair coin, then the probability of a head appearing on one toss is represented as $P(\{H\}) = \frac{1}{2}$.

Example 2
If, on the other hand, our coin is not fair, and we are twice as likely to get a head as we are to get a tail, then $P(\{H\}) = \frac{2}{3}$ and $P(\{T\}) = \frac{1}{3}$.

Group Exercise 1
If a die is rolled and we suppose that all six sides are equally likely to appear, then what is the probability of rolling a "3"? What is the probability of rolling any even number? What is the probability of rolling an even number greater than 3?

The probability that some outcome in the sample space will occur is 1. We have our first axiom of probability:

Theorem 1

For an experiment whose sample space is S, $P(S) = 1$.

There are several ways of determining the probability of an event. The *classical definition of probability* is appropriate when all outcomes of an experiment are equally likely (e.g., the "fair coin" scenario).

Definition 4 (***Classical definition of probability***)

If an experiment has n many possible outcomes, then the classical definition of probability *assigns probability $1/n$ to each of those outcomes. For an event E consisting of k outcomes, the probability of E is given by $P(E) = k/n$.*

Group Exercise 2

What is the probability of drawing a red card (one whose suit is hearts or diamonds) from a standard deck? What is the probability of drawing a face card (jack, queen, or king) of any suit?

$$P(\{\text{red card}\}) = \frac{\#\text{ red cards}}{\#\text{ cards}} = \frac{26}{52} = \frac{1}{2}$$

or

$$P(\{\text{red card}\}) = \frac{\#\text{ red suits}}{\text{total }\#\text{ suits}} = \frac{2}{4} = \frac{1}{2}$$

or

$$P(\{\text{red card}\}) = \frac{1\text{ red color}}{2\text{ total colors}} = \frac{1}{2}.$$

Also, $P(\{\text{face card}\}) = \frac{\#\text{ face cards}}{\#\text{ cards}} = \frac{4 \cdot 3}{52} = \frac{12}{52} = \frac{3}{13}$

or $P(\{\text{face card}\}) = \frac{\#\text{ face cards per suit}}{\#\text{ cards per suit}} = \frac{3}{13}.$

The classical definition is not always appropriate; for example, if we are tossing an unfair coin, then the probability of getting heads is no longer $\frac{1}{2}$. Thankfully, the classical definition is not the only definition of probability.

Definition 5 (***Relative frequency definition of probability***)

If an experiment is performed n times, and if event E occurs f times, then the relative frequency definition of probability *assigns to event E the probability $P(E) = \frac{f}{n}$.*

Example 3

Suppose a bent coin is tossed 50 times and a head appears on 35 of the tosses. The relative frequency definition of probability assigns the probability $35/50 = 0.70$ to the event that a head appears when the coin is tossed.

Of course, the bigger the number of trials (n), the more accurate the relative frequency definition is.

Theorem 2 (***Law of large numbers***)

If a situation, trial, or experiment is repeated again and again, the proportion of occurrences of event E will tend to approach the actual probability of event E.

Group Exercise 3 (***5 minutes***)

For a fair coin, $P(\{H\}) = P(\{T\}) = \frac{1}{2}$. In your group, toss a fair coin 10 times and record the outcomes. Count the heads and count the tails. What probability would the relative frequency definition assign to the event of a head appearing? Toss the coin 10 more times (for a total of 20). What is the relative frequency probability now? Is it closer to the actual value?

For example, 1st 10: $\{H, H, H, T, H, T, T, H, H, T\}$

so $P(\{H\}) = \frac{\#\text{ of H}}{10} = \frac{6}{10}$

and 2nd 10: $\{T, H, T, H, T, T, H, T, H, T\}$

so $P(\{H\}) = \frac{\text{total }\#\text{ of H}}{20} = \frac{10}{20} = \frac{1}{2}.$

Some of you had different results, could indicate a biased coin, a biased flipper, or just a too-small number of trials.

The *subjective definition of probability* differs from the classical and relative definitions.

Definition 6 (*Subjective definition of probability*)

The subjective definition of probability *utilizes intuition, experience, and collective wisdom to assign a probability to an event based on the degree of belief that the event may occur. It allows for several different assignments of probability, which must satisfy* $0 \le P(E) \le 1$ *and* $P(S) = 1$.

Example 4

If a meteorologist reports that there is a 40% chance of rain tomorrow, then (s)he is using the subjective definition of probability to assign the probability of 0.4 to the event that it will rain tomorrow. The classical and relative definitions are not relevant.

Group Exercise 4 (*7 minutes*)

For each of the following probabilities, determine whether the classical, relative, or subjective definitions were used.

1. *When rolling one particular die, the probability of getting a "4" is 1/3.*
2. *A political analyst has told us that the probability of Hillary Clinton's winning the state of Maine in the 2016 general election is 86%.*
3. *When rolling two dice, the probability that the two numbers sum to 7 is 1/6.*
4. *Based on a sample of 1,000 wells in Berkshire County, the probability that a home-owner's water contains unacceptable levels of PFOA is 37%.*

1. Rel. freq.
2. Subjective
3. Cond'l
4. Rel. freq.

There are some simple statements about probability that we can demonstrate, but we first need some terminology.

Definition 7 (*Complement*)

For an event E, *the complement of that event, denoted* E^C, *is everything in the sample space that is not in* E.

Example 5

If S *is the sample space for rolling a 6-sided die, and if* E *is the event of rolling a 5, then we would write* $E := \{5\}$ *and* $E^C = \{1, 2, 3, 4, 6\}$.

Theorem 3

The probability of an event and the probability of its complement sum to 1. That is, for any event E, $P(E) + P(E^C) = 1$.

Group Exercise 5 (*5 minutes*)

For the experiment of rolling two dice, what is the probability of rolling two numbers that do not sum to 7? [Hint: Compute the probability of rolling two numbers that do sum to 7, and use the previous theorem.]

Thm. says: $P(\{\text{sum is not } 7\}) + P(\{\text{sum is } 7\}) = 1$

and by Ex. 4, know $P(\{\text{sum is } 7\}) = \frac{1}{6}$.

So $P(\{\text{sum is not } 7\}) = 1 - \frac{1}{6} = \frac{5}{6}$.

Definition 8 (***Mutual exclusivity***)

Two events E_1 and E_2 are called mutually exclusive *if they cannot occur simultaneously.*

Example 6

When rolling one die, if E_1 is the event that an even number comes up, and if E_2 is the event that an odd number comes up, then E_1 and E_2 are mutually exclusive.

Group Exercise 6 (*5 minutes*)

Suppose that an experiment consists of tossing three fair coins. Write down the sample space for this experiment. Then write down the event E_1 that corresponds to getting two of the same thing in a row (for example, one outcome in this event is (H,H,T) and another is (H,T,T). Write down the event E_2 that consists of all three coin flips having the same result. Are these events mutually exclusive?

$S = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T)\}$.

$E_1 = \{(H,H,H), (H,H,T), (H,T,T), (T,T,H), (T,H,H), (T,T,T)\}$

$E_2 = \{(H,H,H), (T,T,T)\}$. Not mutually exclusive.

Theorem 4

If E_1 and E_2 are two mutually exclusive events, then $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$.

Group Exercise 7

What is the probability of drawing a face card or an ace from a random deck?

There are no cards that are both an ace and a face card, so {ace} and {face card} are mutually exclusive events. Therefore,

$$P(\{\text{ace or face card}\}) = P(\{\text{ace}\}) + P(\{\text{face card}\})$$
$$= \frac{1}{13} + \frac{3}{13} = \frac{4}{13}.$$