L11: 28 Feb. 2017.

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Housekeeping. Homework 4 due Tuesday

webwork due Thursday

Last time: Matrix inverses

This time: ~~More inverse practice?~~
~~LU decomposition?~~
~~Determinants?~~

Linear transformations?

Let $\theta \in [0, 2\pi)$, and consider the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Is it invertible? Find the inv.

$$\left(\begin{array}{cc|cc} \cos\theta & \sin\theta & 1 & 0 \\ -\sin\theta & \cos\theta & 0 & 1 \end{array}\right) \sim \begin{array}{l} R1/\cos\theta \\ R2 + \frac{\sin\theta}{\cos\theta} R1 \end{array} \left(\begin{array}{cc|cc} 1 & \frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & \underbrace{\frac{\sin^2\theta}{\cos\theta} + \cos\theta} & \frac{\sin\theta}{\cos\theta} & 1 \end{array}\right)$$

$$\frac{\sin^2\theta}{\cos\theta} + \cos\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta}$$

$$= \left(\begin{array}{cc|cc} 1 & \frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & \frac{1}{\cos\theta} & \frac{\sin\theta}{\cos\theta} & 1 \end{array}\right) \sim$$

$$= \frac{\cos^2\theta}{\cos\theta} = \cos\theta$$

$$\sim \begin{array}{l} R1 - \sin\theta R2 \\ R2\cos\theta \end{array} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1-\sin^2\theta}{\cos\theta} & -\sin\theta \\ 0 & 1 & \sin\theta & \cos\theta \end{array}\right) =$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & \cos\theta & -\sin\theta \\ 0 & 1 & \sin\theta & \cos\theta \end{array}\right)$$

This is the inverse of $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Note: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, so it is also the transpose of ____

So, the inverse (& transpose) of $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

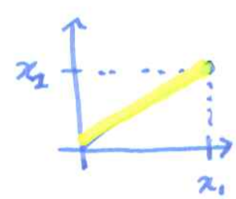
To give more context, a <u>question</u>:

Q: What effect does multiplying a vector $\vec{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ on the 2 by the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ have?

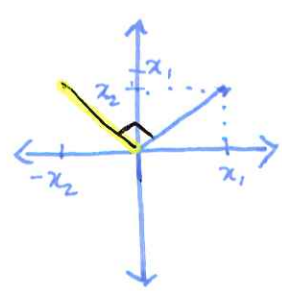
Ans: $$\overbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}^{A(\theta)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{pmatrix}$$

$$2\times 2 \quad 2\times 1 \quad \checkmark$$
$$2\times 1$$

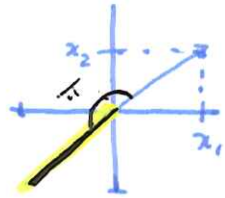
<u>Example</u>: if $\theta = 0$: $A(0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



if $\theta = \frac{\pi}{2}$: $A\left(\frac{\pi}{2}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

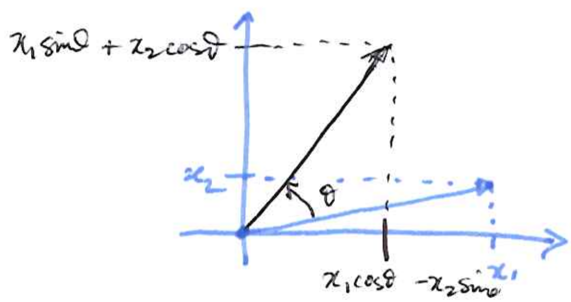


if $\theta = \pi$: $A(\pi)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$



<u>In general</u>: $A(\theta)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{pmatrix}$

is a <u>transformation</u> that rotates the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ abt. the origin through the angle θ.



To "undo" the rotation, you rotate <u>again</u> by $-\theta$.
NEGATIVE

i.e. :

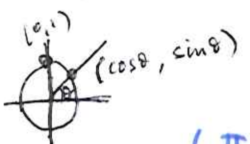
$$\underline{\begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix}}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Well, $\begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Q. What's the matrix that describes the transformation of a vector by rotating it through an angle of 180° abt. the origin?

$$A(\pi) = \begin{pmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

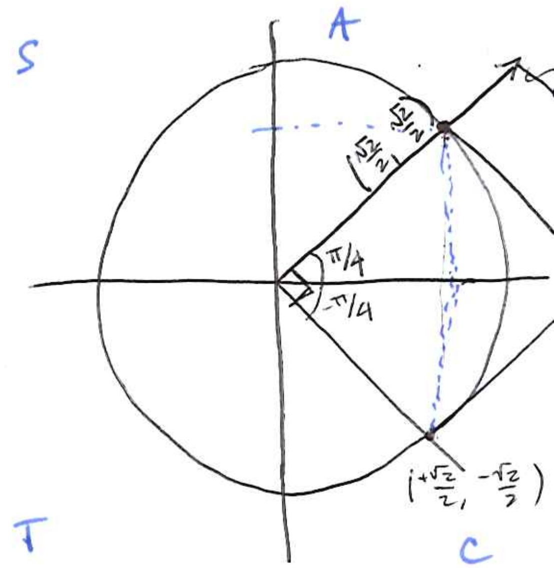
Q. ——— " ——— angle of $\theta = 90°$?



$$A\left(\frac{\pi}{2}\right) = \begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Q. ——— " ——— $\theta = 45°$?

$$A\left(\frac{\pi}{4}\right) = \begin{pmatrix} \cos(\pi/4) & \sin(\pi/4) \\ -\sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$



We just did rotations - a kind of linear transformation (also called a matrix transformation).

- A matrix transformation is a function $f: \mathbb{R}^m \to \mathbb{R}^n$ defined by $f(\vec{x}) = A\vec{x}$, for some matrix $A \in \mathbb{R}^{n \times m}$.

- A linear transformation in $\mathbb{R}^n$ is any function $f: \mathbb{R}^m \to \mathbb{R}^n$ such that for any two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^m$ and for any scalar $a \in \mathbb{R}$,

  - $f(a\vec{x}) = a f(\vec{x})$
  - $f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$.

Claim: Matrix transformations constitute linear transformations.

Pf. Let $A \in \mathbb{R}^{n \times m}$ be fixed, and define $f: \mathbb{R}^m \to \mathbb{R}^n$ by $f(\vec{x}) = A\vec{x}$.

- $$f(a\vec{x}) \underset{\substack{\uparrow \\ \text{def'n of} \\ \text{fn.}}}{=} A(a\vec{x}) \underset{\substack{\uparrow \\ \text{assoc. of} \\ \text{mat. mult.}}}{=} (Aa)\vec{x} \underset{\substack{\uparrow \\ \text{comm. of} \\ \text{mult. of} \\ \text{matrix w/ scalar}}}{=} (aA)\vec{x} \underset{\substack{\uparrow \\ \text{assoc.} \\ \text{of matrix} \\ \text{mult.}}}{=} a(A\vec{x}) \underset{\substack{\uparrow \\ \text{def'n of fn.}}}{=} a f(\vec{x}) \checkmark$$

- $$f(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) \underset{\substack{\uparrow \\ \text{distributive} \\ \text{prop'y}}}{=} A\vec{x} + A\vec{y} = f(\vec{x}) + f(\vec{y}) \checkmark$$

So, matrix transformations are linear transformations.