L12: March 1, 2017.

Housekeeping: • WeBWork due Friday 11:59 p.m.

• " due Tues. 11:59 p.m.

• Written HW — 4 — in class.

Example of a linear transformation:

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y \\ x-z \end{bmatrix}$.

Prove that L is a linear transformation.

(1) $$L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (x_1+x_2)+(y_1+y_2) \\ y_1+y_2 \\ (x_1+x_2)-(z_1+z_2) \end{bmatrix} =$$

$$= \begin{bmatrix} (x_1+y_1)+(x_2+y_2) \\ y_1 + y_2 \\ (x_1-z_1)+(x_2-z_2) \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ y_1 \\ x_1-z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ y_2 \\ x_2-z_2 \end{bmatrix} =$$

$$= L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \checkmark$$

(2) $$L\left(a\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = L\left(\begin{bmatrix}ax\\ay\\az\end{bmatrix}\right) = \begin{bmatrix}ax+ay\\ay\\ax-az\end{bmatrix} = a\begin{bmatrix}x+y\\y\\x-z\end{bmatrix} =$$

$$= aL\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) \checkmark.$$

$$L\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+y\\y\\x-z\end{bmatrix}.$$

$\left\{\begin{array}{l}\text{Can we find a matrix } A \in \mathbb{R}^{3\times 3}\\ \text{s.t. } L(\vec{x}) = A\vec{x} \text{ for all } \vec{x} \in \mathbb{R}^3 ?\end{array}\right.$

Think abt. the action that elementary matrices have ...

$\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}$ stores the op'n $\begin{array}{c}R1+R2\\R2\\R3\end{array}$

$\begin{bmatrix}1&0&0\\0&1&0\\1&0&-1\end{bmatrix}$ —— " —— $\begin{array}{c}R1\\R2\\R1-R3\end{array}$.

We've found a "matrix representation" of L.

This might lead us to the question... do <u>all</u> l.t.'s have a matrix rep'n?

— YES —.

<u>Thm.</u> All linear transformations $\underline{\underline{L : \mathbb{R}^n \to \mathbb{R}^m}}$ have matrix representations.

<u>Pf.</u> Let $\vec{x} \in \mathbb{R}^n$, and let $\vec{x} :\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ where $x_i \in \mathbb{R}$, $i \in [1, n] \cap \mathbb{N}$. By the def'ns of vector add'n & multiplicat'n of vectors by scalars, $\vec{x}$ can be written:

$$\vec{x} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=: \vec{e}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=: \vec{e}_2} + x_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=: \vec{e}_3} + \cdots + x_n \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{=: \vec{e}_n}.$$

$$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3 + x_4 \vec{e}_4 + \cdots + x_n \vec{e}_n.$$

Let ~~Because~~ L be ~~was~~ a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$; then

$$L(\vec{x}) = L(x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n)$$

$$= x_1 L(\vec{e}_1) + x_2 L(\vec{e}_2) + \cdots + x_n L(\vec{e}_n).$$

Let A be the matrix whose i^{th} column is $L(\vec{e}_i)$.

(Note: $A \in \mathbb{R}^{m \times n}$.)

Then

$$A\vec{x} = A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | & & | \\ L(\vec{e}_1) & L(\vec{e}_2) & L(\vec{e}_3) & \cdots & L(\vec{e}_n) \\ | & | & | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 L(\vec{e}_1) + x_2 L(\vec{e}_2) + \cdots + x_n L(\vec{e}_n).$$

$$= L(\vec{x}).$$

So we've found that A is a matrix rep'n. of L.

i.e., $\forall x \in \mathbb{R}^n$, $A\vec{x} = L(\vec{x})$.

So a procedure for finding the matrix rep'n of a l.t. is:

- Transform all elementary vectors in $\mathbb{R}^n$.
- Store the transformations as the columns of A.
- ???
- Profit!

The matrix that accomplishes the matrix rep'n of L is sometimes called the "standard matrix of L".

Ex. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y-z \\ x+z \end{bmatrix}$

The elementary vectors in $\mathbb{R}^3$ are: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1+0 \\ 0-0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0+1 \\ 1-0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0+0 \\ 0-1 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore, $A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ is the std. matrix representing L

Check: $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y-z \\ x+z \end{bmatrix} = L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$. ✓

Question: Is a matrix rep'n of a l.t. unique?

Hypothesis: YES.

Pf. Assume that $L : \mathbb{R}^n \to \mathbb{R}^m$ has rep'n A and has rep'n B.

i.e., $\forall \vec{x} \in \mathbb{R}^n$, $L(\vec{x}) = A\vec{x}$ and $L(\vec{x}) = B\vec{x}$.

~~So, $A\vec{x} = B\vec{x}$ (by transitivity of equality.)~~

So, $A\vec{x} - B\vec{x} = \vec{0}$. i.e., $(A-B)\vec{x} = \vec{0}$. Because $(A-B)\vec{x} = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$, $A-B$ must be the $m \times n$ zero matrix. Since $A - B = 0$, $A = B$.

Exercise: If $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for <u>all</u> x_1 & x_2,

then is it necessarily true that $a_{11} = a_{12} = a_{21} = a_{22} = 0$?

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

so $\boxed{a_{11}x_1 + a_{12}x_2 = 0 = a_{21}x_1 + a_{22}x_2}$, $\forall x_1$ & $x_2 \in \mathbb{R}$.

i.e., $(a_{11} - a_{21})x_1 + (a_{12} - a_{22})x_2 = -a_{21}x_1 - a_{22}x_2 = 0$.