L13: Tuesday, March 7.

Housekeeping: WeBWork due Thursday night

Written homework due Thursday in class

WeBWork + written HW due the week after break.

Exam 2 t/b pushed fwd. by 1wk.

Last time: · Matrix rep'ns of linear transformations

This time: · Different matrix rep'ns?

· Span?

· Linear independence?

Last time, we found the matrix rep'n of a l.t. in $\mathbb{R}^2$ by transforming the <u>elementary vectors</u> $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\end{bmatrix}$.

> This worked because every vector in $\mathbb{R}^2$ can be written as a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\end{bmatrix}$.

Q. : What about other ways?

For example, every vector in $\mathbb{R}^2$ can also be written as a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}1\\1\end{bmatrix}$ ...

... RIGHT?

i.e., Can we always find a & b s.t. for any $\vec{v} \in \mathbb{R}^2$,

$$\vec{v} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix} \quad ?$$

Fix $\vec{v} \doteq \begin{bmatrix}v_1\\v_2\end{bmatrix}$. Want a & b s.t. $a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$.

That is, $a+b=v_1$
$b = v_2$.

If $b=v_2$, and $a = v_1 - v_2$,

then $a+b = (v_1 - v_2) + v_2 = v_1$ ✓ and $b = v_2$ ✓.

Q. What abt. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$? Fix $\vec{v} \in \mathbb{R}^2$.

Can we find a & b s.t. $\vec{v} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

<u>No</u>: $a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and so any vector $\vec{v}$ that's not a scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ cannot be written as $a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

<u>Example</u>: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for any choice of a & b.

<u>Def'n</u>. The set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq V$ is said to <u>span</u> a vector space V if $\forall \vec{v} \in V$, $\exists\, c_1, \dots, c_n$ s.t. $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{v}$.

<u>EX</u>. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^2$.

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ — " — .

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ doesn't span $\mathbb{R}^2$.

Q. Does $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 17 \\ 2 \end{bmatrix} \right\}$ span $\mathbb{R}^2$?

Fix $\vec{v} := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$, and attempt to solve $\vec{v} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 17 \\ 2 \end{bmatrix}$ for a & b.

The vector eq'n is equivalent to the linear system:

$$\begin{cases} a + 17b = v_1 \\ a + 2b = v_2 \end{cases}.$$

Convert to an augmented (?) matrix & row-reduce:

$$\left[\begin{array}{cc|c} 1 & 17 & v_1 \\ 1 & 2 & v_2 \end{array}\right] \sim \begin{matrix} R1 \\ -R2+R1 \end{matrix} \underbrace{\left[\begin{array}{cc|c} \textcircled{1} & 17 & v_1 \\ 0 & \textcircled{15} & v_1 - v_2 \end{array}\right]}_{\text{REF}} \sim$$

$$\sim \begin{matrix} R1 \\ \frac{1}{15} R2 \end{matrix} \left[\begin{array}{cc|c} 1 & 17 & v_1 \\ 0 & 1 & \frac{v_1 - v_2}{15} \end{array}\right]$$

- It's enough to stop at REF (going to RREF involves more arithmetic + no add'l benefit, as we cared only abt. whether a sol'n existed, not abt. what the sol'n was).
- If there are no ~~free variables~~ — i.e., no rows of zeros — then, yes, a sol'n exists for any $\vec{v}$.

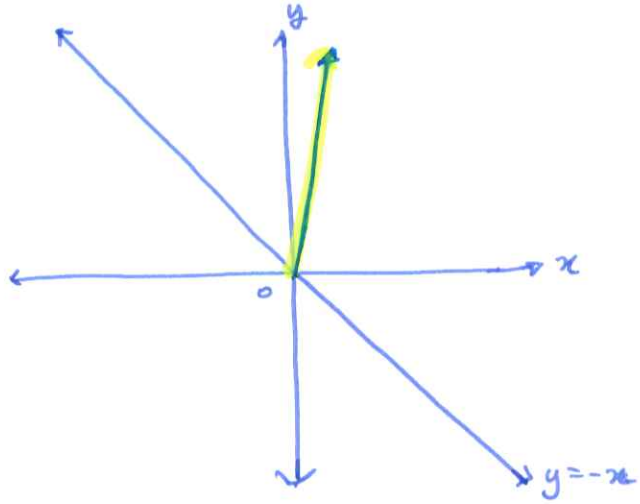
Q. Does $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ span $\mathbb{R}^2$?

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} \sim \begin{matrix} R1 \\ R2-3R1 \end{matrix} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -2 \end{bmatrix}}$$

REF:
- rows of 0 at btm
- 0's below pivots
- pivots to (R) of pivots above

For this system, there are no rows of zeros; there's a free variable (c_3), but that doesn't mean there are no sol'ns, just that there are infinitely many.

Reflec'n abt. $y = -x$:



Q. Can we redefine "rotat'n" and "rotat'n matrices" t/mean only clockwise rotat'n?

Thoughts :
- rotating $\vec{v} \in \mathbb{R}^2$ abt. the origin through an angle of θ in the clockwise direc'n is the same as rotating ccw by $2\pi - \theta$ or $360^\circ - \theta$.
- if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $T(\vec{v})$ rotates $\vec{v}$ ccw through an angle of θ, then
$$T(\vec{v}) = A\vec{v}, \text{ where } A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Q. If $U : \mathbb{R}^2 \to \mathbb{R}^2$ is s.t. $U(\vec{v})$ rotates $\vec{v}$ cw through θ, then what is A s.t. $U(\vec{v}) = A\vec{v}$ $\forall \vec{v}$?

$$u_\theta(\vec{v}) = T_{2\pi-\theta}(\vec{v}) = \begin{pmatrix} \cos(2\pi-\theta) & \sin(2\pi-\theta) \\ -\sin(2\pi-\theta) & \cos(2\pi-\theta) \end{pmatrix} \vec{v}$$

$$= \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{A^T = A^{-1}} \vec{v}$$

Example.

(a)

$$\begin{bmatrix} 5 & 7 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5(1) - 1(7) \\ 6(1) - 1(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

(2×2, 2×1)

$$1\begin{bmatrix} 5 \\ 6 \end{bmatrix} - 1\begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 - 1(7) \\ 6 - 1(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 5 & 7 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 5\cdot 1 + 7(-1) & 5\cdot 0 + 7\cdot 3 \\ 6\cdot 1 + 0(-1) & 6\cdot 0 + 0\cdot 3 \end{bmatrix}$$

(2×2, 2×2)

(c)

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_m \\ | & | & & | \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = c_1 a_1 + c_2 a_2 + c_3 a_3 + \cdots + c_m a_m$$

(d)

$$\begin{bmatrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{bmatrix}\begin{bmatrix} c_{11} & \cdots & c_{1m} \\ c_{21} & \cdots & c_{2m} \\ c_{31} & \cdots & \vdots \\ \vdots & \ddots & \\ c_{m1} & \cdots & c_{mm} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ c_{11}a_1 + \cdots + c_{m1}a_m & c_{12}a_1 + \cdots + c_{m2}a_m & \cdots & c_{1m}a_1 + \cdots + c_{mm}a_m \\ | & | & & | \end{bmatrix}$$