Another use of the word <u>span</u> is the noun usage (we had "span" as a vb. before). The noun:

Given a set of vectors $S := \{v_1, v_2, \dots, v_m\}$, the <u>span of S</u> is the ~~vector~~ set of vectors:

$$\text{span}(S) := \{v : v = a_1v_1 + a_2v_2 + \dots + a_mv_m, \text{ for some } a_i,\ i \in [1, m] \cap \mathbb{N}\}.$$

<u>Example</u>. $S = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$, then

$$\text{span}(S) = \left\{ v : v = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} \right\} = \left\{ v : v = \begin{bmatrix}a\\b\end{bmatrix} \right\} = \mathbb{R}^2.$$

<u>Example</u>. What is the span of $S := \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\end{bmatrix} \right\}$?

$$\text{span}(S) = \left\{ v : v = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\0\\2\end{bmatrix} \right\} = \left\{ v : v = \begin{bmatrix}a\\0\\2b\end{bmatrix},\ a, b \in \mathbb{R} \right\} =$$

$$= \left\{ v : v = \begin{bmatrix}a\\0\\b\end{bmatrix},\ a, b \in \mathbb{R} \right\} = \text{the } xz\text{-plane of } \mathbb{R}^3.$$

L14, ct'd.

Example. $S := \{x, 1, x^3\}$.

$$\text{span}(S) = \{v : v = ax + b\cdot 1 + c\cdot x^3\}$$
$$= \{v : v = cx^3 + ax + b \;,\; a, b, c \in \mathbb{R}\}$$

Q. Is $p(x) := x^3 - 2x^2 + 7$ in $\text{span}(S)$?

i.e., $\overset{?}{\exists}\, a, b, c \in \mathbb{R}$ s.t. $p(x) = cx^3 + ax + b$

NO : the presence of $-2x^2$ in $p(x)$ makes this impossible!

Q. Is $q(x) := 3x^3 - x - 4$ in $\text{span}(S)$?

i.e., $\overset{?}{\exists}\, a, b, c \in \mathbb{R}$ s.t. $3x^3 - x - 4 = cx^3 + ax + b$

$$\left.\begin{aligned} a &= -1 \\ b &= -4 \\ c &= 3 \end{aligned}\right\} \text{ yields } \quad cx^3 + ax + b = 3x^3 - x - 4 = q(x) \checkmark$$

So, YES, $q(x) \in \text{span}(S)$.

Q. Is $r(x) := 17x \in \text{span}(S)$? $\quad S = \{x, 1, x^3\}$.

$$r(x) = \underbrace{0}_{\hookrightarrow c=0}\, x^3 + \underbrace{17}_{\hookrightarrow a=17} x + \underbrace{0}_{\hookrightarrow b=0} \cdot 1$$

Example. Find $\operatorname{span}\Big(\overbrace{\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}\right\}}^{=:S}\Big)$.

$$\operatorname{span}(S) = \left\{ v : v = a\begin{bmatrix}1&0\\0&0\end{bmatrix} + b\begin{bmatrix}0&1\\0&0\end{bmatrix} + c\begin{bmatrix}0&0\\1&1\end{bmatrix}, \text{ some } a,b,c\in\mathbb{R}\right\}$$

$$= \left\{ v : v = \begin{bmatrix}a&b\\c&c\end{bmatrix}, \text{ some } a,b,c\in\mathbb{R}\right\}.$$

Vector spaces.

A vector space is a set V of objects, called vectors, together with a ~~set~~ field F of other objects, called scalars, and with the operation $\oplus$ and the operation $\odot$, s.t. the following properties hold:

1. V is closed under addition, i.e., if $\vec{v}\in V$ and $\vec{w}\in V$, $\vec{v}\oplus\vec{w}\in V$.
   - $\vec{v}\oplus\vec{w} = \vec{w}\oplus\vec{v} \quad \forall \vec{v},\vec{w}\in V$ commutativity
   - associativity: $\vec{u}\oplus(\vec{v}\oplus\vec{w}) = (\vec{u}\oplus\vec{v})\oplus\vec{w} \quad \forall \vec{u},\vec{v},\vec{w}\in V$
   - $\exists\, \vec{0}\in V$ s.t. $\vec{0}\oplus\vec{v} = \vec{v} \quad \forall \vec{v}\in V$.
   - $\forall \vec{v}\in V$, $\exists\, \vec{w}\in V$ s.t. $\vec{v}\oplus\vec{w} = \vec{0}$.

(2) V is closed under scalar multiplication, i.e.,

if $\vec{v} \in V$ and $c \in F$, then $c \odot \vec{v} \in V$.

- distributivity: $c \odot (\vec{u} \oplus \vec{v}) = c \odot \vec{u} \oplus c \odot \vec{v}$ $\forall \vec{u}, \vec{v} \in V$ $\forall c \in F$
- — " — $(c+d) \odot \vec{v} = c \odot \vec{v} \oplus d \odot \vec{v}$ $\forall \vec{v} \in V$, $\forall c, d \in F$
- "associativity" $(c \cdot d) \odot \vec{v} = c \odot (d \odot \vec{v})$ $\forall \vec{v} \in V$, $\forall c, d \in F$.
- identity: $\exists c \in F$ s.t. $c \odot \vec{u} = \vec{u}$ $\forall \vec{u} \in V$.

There are 10 things to check !!

Example. Verify that $\mathbb{R}^4$ is a vector space over $\mathbb{R}$.

Assume $\vec{u} := \langle u_1, u_2, u_3, u_4 \rangle$ and $\vec{v} := \langle v_1, v_2, v_3, v_4 \rangle$ $\vec{w} := \langle w_1, w_2, w_3, w_4 \rangle$

are in $\mathbb{R}^4$, and that $a, b \in \mathbb{R}$.

(1) $\vec{u} \oplus \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4 \rangle \in \mathbb{R}^4$. ✓

- $\vec{u} \oplus \vec{v} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3, v_4 + u_4 \rangle = \vec{v} \oplus \vec{u}$. ✓
- $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = \langle u_1, u_2, u_3, u_4 \rangle + \langle v_1 + w_1, v_2 + w_2, v_3 + w_3, v_4 + w_4 \rangle$

$$= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3), u_4 + (v_4 + w_4) \rangle$$
$$= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3, (u_4 + v_4) + w_4 \rangle$$
$$= \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4 \rangle + \langle w_1, w_2, w_3, w_4 \rangle$$
$$= (\vec{u} \oplus \vec{v}) \oplus \vec{w}. \checkmark$$

- $\overset{?}{\exists}\, \vec{0} \in \mathbb{R}^4$. Try $\langle 0,0,0,0 \rangle$: ✓

$$\vec{u} \oplus \vec{0} = \langle u_1+0, u_2+0, u_3+0, u_4+0 \rangle = \langle u_1, u_2, u_3, u_4 \rangle = \vec{u}$$

- Let $\vec{z} := \langle -u_1, -u_2, -u_3, -u_4 \rangle$.

Then $\vec{u} \oplus \vec{z} = \langle u_1-u_1, u_2-u_2, u_3-u_3, u_4-u_4 \rangle$
$= \langle 0,0,0,0 \rangle$
$= \vec{0}$. ✓

(2) $a \odot \vec{v} = \langle av_1, av_2, av_3, av_4 \rangle \in \mathbb{R}^4$. ✓

- $a \odot (\vec{v} \oplus \vec{w}) = a \odot \langle v_1+w_1, v_2+w_2, v_3+w_3, v_4+w_4 \rangle$
$= \langle a(v_1+w_1), a(v_2+w_2), a(v_3+w_3), a(v_4+w_4) \rangle$.
$= \langle av_1+aw_1, av_2+aw_2, av_3+aw_3, av_4+aw_4 \rangle$
$= a \odot \vec{v} \oplus a \odot \vec{w}$. ✓

- $(c+d) \odot \vec{v} = \langle (c+d)v_1, (c+d)v_2, (c+d)v_3, (c+d)v_4 \rangle$
$= \langle cv_1+dv_1, cv_2+dv_2, cv_3+dv_3, cv_4+dv_4 \rangle$
$= \langle cv_1, cv_2, cv_3, cv_4 \rangle + \langle dv_1, dv_2, dv_3, dv_4 \rangle$
$= c \odot \vec{v} \oplus d \odot \vec{v}$ ✓

- $(c \cdot d) \odot \vec{v} = \langle (cd)v_1, (cd)v_2, (cd)v_3, (cd)v_4 \rangle$
$= \langle c(dv_1), c(dv_2), c(dv_3), c(dv_4) \rangle$
$= c \odot (d \odot \vec{v})$ ✓

- $\overset{?}{\exists}\, c \in \mathbb{R}$ s.t. $c \odot \vec{u} = \vec{u}$ $\forall \vec{u} \in \mathbb{R}^4$? Yes; $1 \odot \vec{u} = \langle 1u_1, 1u_2, 1u_3, 1u_4 \rangle$
$= \langle u_1, u_2, u_3, u_4 \rangle = \vec{u}$ ✓

* Vector subspaces.

If V is a vector space and $W \subseteq V$, and $W \neq \{\}$, then W is said t/b a <u>subspace</u> of V if W is a vector space in and of itself, w.r.t. the same scalar field and operations under which V was a vector space.

<u>Note</u>: As V was a v.s., W "inherits" 8 of 10 properties <u>FOR 'FREE'</u> – need only to prove:

- Closure under add'n
- —— " —— scalar multiplicat'n.