L15: March 21, 2017.

Last time: Span

Vector spaces

Questions?

This time: Vector spaces

Vector subspaces

Some common vector spaces...

$\mathbb{R}^n$, all n-dim'l vectors w. real components

$\mathbb{R}^{m \times n}$, all $m \times n$ matrices w. real entries

$F[a,b]$, all real-valued fns. defined on $[a,b]$

$F(-\infty, \infty)$, —— " —————— $(-\infty, \infty)$

P_n, all polynomials of degree $\leq n$

P, all polynomials

} with standard notions of $\oplus$ and $\odot$, and over the scalar field of real #s

Recall: there are 10 vector space axioms to show for each one!

We showed $\mathbb{R}^3$ was a vector space... let's assume the others all are, too.

A <u>vector subspace</u> of a v.s. V is a nonempty ~~st~~ subset $W \subseteq V$ that is, itself, a v.s. w.r.t. the operat'ns on V and w.r.t. the scalar field of V.

<u>Note:</u> All but two of the axioms for a v.s. come for "free", just because everything in W came from V to begin with. All that's left to prove:

① Closure under $\oplus$

② Closure under $\odot$

<u>Example.</u> $\mathbb{Z}$ do <u>not</u> constitute a subspace of $\mathbb{R}$, because $\mathbb{Z}$ is not closed under scalar multiplication:

$$2 \in \mathbb{Z}, \quad \tfrac{1}{3} \in \mathbb{R}, \quad \text{but} \quad \tfrac{1}{3} \cdot 2 = \tfrac{2}{3} \notin \mathbb{Z}.$$

<u>Example.</u> $\{0\} \subseteq \mathbb{R}$ is a vector subspace of $\mathbb{R}$, because:

① $0 + 0 = 0 \in \{0\}$. ✓

② $\forall c \in \mathbb{R}$, $c \cdot 0 = 0 \in \{0\}$. ✓

The remaining 8 v.s. axioms ~~as~~ are inherited from ~~the fact~~ $\mathbb{R}$ because it was a v.s.

L15, ct'd.

Any given vector space V always has two subspaces:

- $\{\vec{0}\}$, as $\vec{0} \oplus \vec{0} = \vec{0}$ and $c \odot \vec{0} = \vec{0} \quad \forall c \in \mathbb{R}$ } v.s. axioms

- V itself.

Q: Why does the existence of $\vec{0}$ in a vector subspace W follow from W's closure under $\oplus$ and under $\odot$, if $W \subseteq V$, and V is a v.s.?

Pf. If W is closed under scalar mult., and if $\vec{x} \in W$, then $-1 \cdot \vec{x} \in W$.

If W is closed under vector add'n, and if $\vec{x} \in W$ and $-1 \cdot \vec{x} \in W$, then

$$\vec{x} + (-1 \cdot \vec{x}) \in W.$$

Since $\vec{x} \in W$, $\vec{x} \in V$, and as V was a v.s., then distributivity holds:

$$\begin{aligned} \vec{x} + (-1)\vec{x} &= (1 + (-1))\vec{x} \in W \\ &= 0 \cdot \vec{x} \in W \\ &= \vec{0} \in W. \end{aligned}$$

by the v.s. axiom. □

Suppose V is a vector space, and $W \subseteq V$, and W is closed under $\oplus$ and $\odot$. Prove $\vec{0} \in W$.

Pf. Since W is closed under $\odot$, and since $0 \in \mathbb{R}$, then if $\vec{x} \in W$, $0 \cdot \vec{x} \in W$. Because $\vec{x} \in W$, $\vec{x} \in V$, and so, because V was a v.s., $0 \cdot \vec{x} = \vec{0}$. Therefore since $0 \cdot \vec{x} \in W$, $\vec{0} \in W$. $\square$

Example[3]. Consider $W \subseteq \mathbb{R}^{2 \times 3}$, where

$$W = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}, \text{ some } a, b, c, d \in \mathbb{R} \right\}.$$

Closure under $\oplus$:

Let $\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}$ and $\begin{bmatrix} e & f & 0 \\ 0 & g & h \end{bmatrix}$ be in W (i.e., let $a, b, \ldots, h \in \mathbb{R}$).

Observe: $$\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix} + \begin{bmatrix} e & f & 0 \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} (a+e) & (b+f) & 0 \\ 0 & (c+g) & (d+h) \end{bmatrix}.$$

Further, observe that since $a, b, \ldots, h \in \mathbb{R}$,

$$(a+e) \in \mathbb{R}, \quad (b+f) \in \mathbb{R}, \quad (c+g) \in \mathbb{R}, \quad (d+h) \in \mathbb{R}.$$

So $\begin{bmatrix} a+e & b+f & 0 \\ 0 & c+g & d+h \end{bmatrix} \in W$. ✓

Closure under $\odot$: Let $a, b, c, d \in \mathbb{R}$, and let $steve \in \mathbb{R}$.

$$steve \cdot \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix} = \begin{bmatrix} a \cdot steve & b \cdot steve & 0 \\ 0 & c \cdot steve & d \cdot steve \end{bmatrix},$$

and since $a, \ldots, d \in \mathbb{R}$ and $steve \in \mathbb{R}$,

$a \cdot steve$, $b \cdot steve$, $c \cdot steve$, and $d \cdot steve \in \mathbb{R}$, so

$$\begin{bmatrix} a \cdot steve & b \cdot steve & 0 \\ 0 & c \cdot steve & d \cdot steve \end{bmatrix} \in W.$$

So, since W was closed under $\oplus$ & under $\odot$,

and since $W \subseteq \mathbb{R}^{2 \times 3}$ & $\mathbb{R}^{2 \times 3}$ is a v.s., W is

a vector subspace of $\mathbb{R}^{2 \times 3}$.

Ex. 4 $W \subseteq \mathbb{R}^{2\times 3}$, $W = \left\{ \begin{bmatrix} 0 & a & b \\ c & d & 1 \end{bmatrix}, \ a,b,c,d \in \mathbb{R} \right\}$.

W is not a v. subsp. of $\mathbb{R}^{2\times 3}$, since it fails t/b closed under ⊕. In particular,

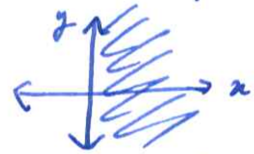
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in W$$

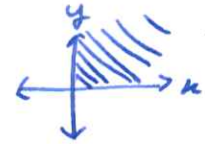
and $$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in W,$$

OH NO!

but $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \notin W$$

Examples. Which of the following are v. subspaces of $\mathbb{R}^2$?

(a) $W_1 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{matrix} x \geq 0 \\ y \in \mathbb{R} \end{matrix} \right\}$ 

(b) $W_2 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, x \geq 0 \text{ \& } y \geq 0 \right\}$ 

(c) $W_3 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{matrix} x = 0 \\ y \in \mathbb{R} \end{matrix} \right\}$ 

For each, sketch W_i on the Cartesian coord. plane.

(a) W_1 Fails t/b closed under $\odot$:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W_1 \text{ and } -2 \in \mathbb{R} \text{ but } -2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \notin W_1.$$

(b) W_2 fails t/b closed under $\odot$:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W_2, \quad -2 \in \mathbb{R}, \quad -2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \notin W_2.$$

(c) Let $\begin{bmatrix} 0 \\ y \end{bmatrix} \in W_3$, let $\begin{bmatrix} 0 \\ z \end{bmatrix} \in W_3$. Then

$$\begin{bmatrix} 0 \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y+z \end{bmatrix} \in W_3, \text{ since } y, z \in \mathbb{R} \Rightarrow y+z \in \mathbb{R}.$$

Let $\begin{bmatrix} 0 \\ y \end{bmatrix} \in W_3$, $c \in \mathbb{R}$. Then $c\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ cy \end{bmatrix} \in W_3$,

since $\left. \begin{matrix} y \in \mathbb{R} \\ c \in \mathbb{R} \end{matrix} \right\} \Rightarrow cy \in \mathbb{R}$.

Ex. 5. P_3 is the v.s. of polynomials of degree 3 or less.

$$P_3 = \{ c_0 + c_1 x + c_2 x^2 + c_3 x^3, \quad c_0, \dots, c_3 \in \mathbb{R} \}.$$

Prove that $P_2 := \{ c_0 + c_1 x + c_2 x^2, \ c_0, c_1, c_2 \in \mathbb{R} \}$ is a v. subsp. of P_3.

Let $a + bx + cx^2 \in P_2$, $d + ex + fx^2 \in P_2$. Then

$(a+bx+cx^2) + (d+ex+fx^2) = (a+d) + (b+e)x + (c+f)x^2 \in P_2$,

since all coffs. are in $\mathbb{R}$.

Let $a+bx+cx^2 \in P_2$, let $s \in \mathbb{R}$. Then

$s(a+bx+cx^2) = sa + s \cdot bx + s \cdot c\, x^2 \in P_2$, since all coffs. are in $\mathbb{R}$.

Example 6. Let V be the space of polynomials of exactly degree 2. Is V a v. subsp. of P_2?

Fails t/b closed under $\oplus$:

$x^2 + x + 1 \in V$, and $-x^2 + 3x + 0 \in V$.

But $(x^2+x+1) + (-x^2+3x+0) = 4x + 1 \notin V$.

Example. $C[a,b]$ denotes the set of all real-valued fns. on $[a,b]$ tht. are cts.

Is $C[a,b]$ a v. subsp. of $F[a,b]$?

If $f \in C[a,b]$ and $g \in C[a,b]$, is $f+g \in C[a,b]$?

[**Continuity**: f is cts. at $c \in [a,b)$ if $f(c) = \lim_{x\to c} f(x)$

— " — on $[a,b]$ if f is cts. at c, $\forall c \in [a,b]$]

→ Yes, by the rules of limits & the def'n of continuity:

if $f(c) = \lim_{x\to c} f(x)$ and $g(c) = \lim_{x\to c} g(x)$,

then
$$\underbrace{f(c)+g(c)}_{\displaystyle \overset{\shortparallel}{(f+g)(c)}} = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \lim_{x\to c} f(x)+g(x) = \lim_{x\to c} (f+g)(x)$$

✓ $(f+g)(c) = \lim_{x\to c}(f+g)(x)$

11

Also for $\odot$:
$$s\cdot f(c) = s\cdot \lim_{x\to c} f(x) = \lim_{x\to c} s\cdot f(x) = \lim_{x\to c} (sf)(x)$$

$$s\cdot f(c) = (sf)(c)$$

$(sf)(c) = \lim_{x\to c}(sf)(x)$