L17 : March 28, 2017 (Tuesday)

Housekeeping : • Exam 2 will be take-home (due Tuesday in class)

• Webwork due Thursday 11:59 pm

Last time : Row space

Column space

This time : Basis

Dimension

L14.pdf — modelling.

WW2.

$V = \mathbb{R}^{2\times 2}$, $H := \{ M \in \mathbb{R}^{2\times 2} : M \cdot M = M \}$

① $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \overset{?}{\in} H$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \checkmark$$

② Is H closed under add'n?

Assume $A \in H$ and $B \in H$. Is $(A+B) \overset{?}{\in} H$?

Since $A \in H$, $A \cdot A = A$; since $B \in H$, $B \cdot B = B$.

Now $$(A+B)(A+B) = A\cdot A + \underbrace{A\cdot B + B \cdot A}_{\neq 2A\cdot B} + B \cdot B$$

$$= A + B + \underbrace{(A\cdot B + B \cdot A)}_{\overset{?}{=} 0},$$

no – in general, $(A+B)(A+B) \neq A+B$, so $A+B \notin H$ necessarily.

② ct'd. Note, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$, because

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$,

and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$,

so $A := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

give $(A+B) \notin H$.

Try: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ✓

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} aa + bc & ab + bd \\ ac + cd & bc + dd \end{pmatrix}$$

want: $a^2 + bc = a$

$ab + bd = b \Rightarrow a + d = b$

$ac + cd = c \Rightarrow a + d = c$

$bc + d^2 = d$

L17, cty.

$P_2 = \{ax^2 + bx + c;\ a, b, c \in \mathbb{R}\}$

A) $A = \{p(x) : p(4) = 0\}$.

① Closed under add'n?

Assume $p(x) = ax^2 + bx + c$, $p(4) = 0$

$q(x) = dx^2 + ex + f$, and $q(4) = 0$

$(p+q)(4) = p(4) + q(4) = 0 + 0 = 0$,

~~Linear fns: $f(a+b) = f(a) + f(b)$~~

~~$f(c \cdot a) = c \cdot f(a)$~~

so $(p+q) \in A$.

② Closed under scal. mult.?

Assume $p(x) \in A$, so $p(4) = 0$.

$(cp)(4) = c \cdot p(4) = c \cdot 0 = 0$, so $(cp)(x) \in A$.

YES

$C = \{ p(x) : p(6) = 8 \}$. NOOO

$D = \{ p(x) : \int_0^8 p(t)\, dt = 0 \}$

① Add'n: Assume $p, q \in D$. $\int_0^8 (p+q)(t)\, dt = \int_0^8 p(t)\, dt + \int_0^8 q(t)\, dt$
$= 0 + 0 = 0$ ✓

② Scal. mult? Ass. $p \in D$. $\int_0^8 c\,p(t)\, dt = c\int_0^8 p(t)\, dt = c \cdot 0 = 0$ ✓

$E = \{ p(t) : p'(t) + 7p(t) + 6 = 0 \}$

$0 \notin E$, bc. $(0')(t) + 7 \cdot (0)(t) + 6 = 0 + 0 + 6 = 6 \neq 0$

$F \doteq \{ p(t) : p(-t) = p(t) \ \forall t \}$. "Even functions"

① Add'n: $p(t) \in F$, $q(t) \in F$. So $p(-t) = p(t)$ and $q(-t) = q(t)$.

Check: $(p+q)(-t) = p(-t) + q(-t) = p(t) + q(t) = (p+q)(t)$ ✓

② Mult: $p(t) \in F$, $c \in \mathbb{R}$. So $p(-t) = p(t)$.

check: $(cp)(-t) = c \cdot p(-t) = c\,p(t) = (cp)(t)$ ✓

$B := \{ p(x) : p'(x) \text{ is constant} \}$.

① Assume $p(x) \in B$, $q(x) \in B$. So $p'(x) = c_1$ and $q'(x) = c_2$, some $c_1, c_2 \in \mathbb{R}$.

Is $(p+q)(x) \in B$?

$(p+q)' = p' + q' = \underbrace{c_1 + c_2}_{\text{constant}}$, so, yes, $(p+q) \in B$.

② Assume $p(x) \in B$, $c \in \mathbb{R}$. So $p'(x) = c_1 \in \mathbb{R}$.

Is $(cp)(x) \in B$?

$(cp)' = c \cdot p' = \underbrace{c \cdot c_1}_{\text{const.}}$ so, $cp \in B$.

L17, ct'd.

①
Assume $p(x) = ax^3 + bx$, $q(x) = cx^3 + dx$, $a, b, c, d \in \mathbb{R}$.

$$(p+q)(x) = p(x) + q(x) = ax^3 + bx + cx^3 + dx$$
$$= \underbrace{(a+c)}_{\in \mathbb{R}} x^3 + \underbrace{(b+d)}_{\in \mathbb{R}} x. \quad \checkmark$$

② $p(x) = ax^3 + bx$, $a, b, c \in \mathbb{R}$.

$$(cp)(x) = c \cdot p(x) = c(ax^3 + bx) = \underbrace{(ac)}_{\in \mathbb{R}} x^3 + \underbrace{(cb)}_{\in \mathbb{R}} x \quad \checkmark$$

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① $\begin{bmatrix} x \\ y \end{bmatrix} \in S$, so $x^2 - y^2 = 0$

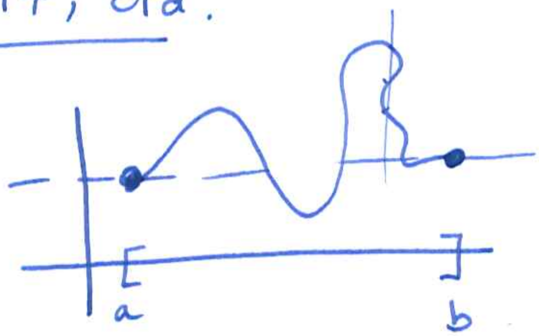
$\begin{bmatrix} z \\ w \end{bmatrix} \in S$, so $z^2 - w^2 = 0$.

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} x+z \\ y+w \end{bmatrix} \overset{?}{\in} S$$

$$(x+z)^2 - (y+w)^2 \overset{?}{=} 0$$
$$= x^2 + 2xz + z^2 - y^2 - 2yw - w^2$$
$$= (x^2 - y^2) + (z^2 - w^2) + 2xz - 2yw$$
$$= 2xz - 2yw \neq 0 \text{ necessarily.}$$

So S fails closure under vec. add'n.

L17, ctd.



$S = \{ f : f(a) = f(b) \}$.

① Add'n: Ass. $\left.\begin{array}{l} f(a) = f(b) \\ g(a) = g(b) \end{array}\right\} \longrightarrow$

$\boxed{\text{check}: (f+g)(a) \overset{?}{=} (f+g)(b)}$

$(f+g)(a) = f(a) + g(a)$
$= f(b) + g(b)$
$= (f+g)(b)$ ✓

② Mult. Assume $f(a) = f(b)$, $c \in \mathbb{R}$.

$\boxed{\text{Check}: (cf)(a) \overset{?}{=} (cf)(b)}$

$(cf)(a) = c \cdot f(a) = c \cdot f(b)$ ✓
$= (cf)(b)$

$S = \{ M \in \mathbb{R}^{n \times n} : M = M^T \}$

① $\S \left\{\begin{array}{l} A = A^T \\ B = B^T \end{array}\right.$.

Check: $(A+B)^T \overset{?}{=} (A+B)$

$(A+B)^T = A^T + B^T = A + B$ ✓

② $\S$ $A = A^T$, $c \in \mathbb{R}$.

check: $(cA)^T \overset{?}{=} cA$

$(cA)^T = c \cdot A^T = c \cdot A$ ✓