L18, March 30, 2017.

House keeping: - WW due 11:59 p.m. tonight

- Exam 2 due in class Tuesday

Last time: Review of vector spaces & subspaces

This time: Basis + Dimension

Recall: $S := \{\vec{v}_1, \dots, \vec{v}_m\} \subseteq V$ (V a vector space) is said to SPAN V if:

$\forall \vec{v} \in V$, $\exists c_1, \dots, c_m \in \mathbb{R}$ s.t. $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m$.

If there are a lot of vectors in S, then S is more likely to span V.

e.g.: $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ doesn't span $\mathbb{R}^3$:

$\begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and the absence of a pivot in the 3^{rd} row means there are some vectors $\langle a, b, c \rangle$ for which $d\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has no solution!

$$\left[\begin{array}{cc|c} 3 & 0 & a \\ 0 & 0 & b \\ 1 & 1 & c \end{array}\right] \sim \begin{array}{c} \frac{R1}{3} \\ R3-\frac{1}{3}R1 \\ R2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & a/3 \\ 0 & 1 & c - a/3 \\ 0 & 0 & b \end{array}\right]$$

$\leftarrow$ if $b \neq 0$, then $\bar{\exists}$ sol'n.

e.g.: $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$:

$$\begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{REF — pivots in all rows.}}$$

<u>Recall</u>: $S := \{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly dependent if

$\exists\, c_1, \dots, c_n \in \mathbb{R}$ <u>not all zero</u> s.t. $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$.

S is linearly independent if it is not linearly dependent.

e.g.: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Solve for c_1, c_2:

$$c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

If the only sol'n is $c_1 = c_2 = 0$, then <u>dep.</u>; otherwise indep. — So row reduce:

$$\begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{array}{c} R2 \\ R1+R2 \\ R3 \\ R4 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & -2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{array}{c} R1 \\ R3 \\ R4-R3 \\ R2+2R3 \end{array} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{REF.}}.$$

There is a pivot in every <u>col</u>, so no free variables!

Only sol'n is trivial so <u>indep</u>.

e.g.: $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ +1 \end{bmatrix} \right\}$.

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & -2 & 2 \\ -1 & 1 & +1 \end{bmatrix} \sim \begin{array}{c} R1 \\ R2-2R1 \\ R3+R1 \end{array} \begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{array}{c} R1 \\ R2/-4 \\ R3 - \frac{2R2}{-4} \\ = R3 + \left(\frac{1}{2}\right)R2 \end{array} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Yes - these vectors <u>are</u> lin. indep. (Pivot in ea. col.)

e.g.: $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & -2 & 2 \\ -1 & 1 & -1 \end{bmatrix} \sim \begin{array}{c} R1 \\ R2-2R1 \\ R3+R1 \end{array} \begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 2 & -4 \end{bmatrix} \sim \begin{array}{c} R1 \\ R2/{-4} \\ R3+\frac{R2}{2} \end{array} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

no pivot

if trying to solve

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} :$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 2 & -2 & 2 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \underbrace{\begin{array}{c} R1-R2 \\ R2 \\ R3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{RREF}}$$

c_3 is a free variable

$c_2 = 2c_3$

$c_1 = c_3$.

so choose $c_3 \in \mathbb{R}$. then $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a non-triv. soln to the system above.

$1 \cdot c_1 + 0 \cdot c_2 - 1 \cdot c_3 = 0 \iff c_1 - c_3 = 0 \iff \boxed{c_1 = c_3}$

i.e., $c_3 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + 2c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$c_3 \underbrace{\left(\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} \right)}_{\langle 0, 0, 0 \rangle} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The more vectors in a set, the <u>less likely</u> to be linearly independent.

Def. A <u>BASIS</u> for a vector space V is a ~~subspace~~ subset $S \subseteq V$ that spans V <u>and</u> is linearly independent.

<u>examples.</u> $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$. Is it a basis?

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Pivot in every <u>row</u> : <u>spans $\mathbb{R}^2$</u>.
- Pivot in every <u>col</u> : <u>linearly indep.</u>

This particular basis of $\mathbb{R}^2$ is called an <u>ORTHONORMAL</u> basis of $\mathbb{R}^2$.

- Two vectors are orthogonal if their dot prod. is zero.
- Two vectors are "normal" if their dot prod. is 1.

A set S is <u>orthonormal</u> if $\forall v \in S$,

$$\left.\begin{aligned} \vec{v} \cdot \vec{w} &= 0 \quad \text{if } \vec{v} \neq \vec{w} \\ \vec{v} \cdot \vec{v} &= 1. \end{aligned}\right\}$$

"all vectors are normal to themselves; orthogonal to all others".

Example. $S := \left\{ \begin{bmatrix} 17 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$.

$$\begin{bmatrix} 17 & 3 \\ 1 & 2 \end{bmatrix} \sim \begin{array}{c} R2 \\ R1-17R2 \end{array} \begin{bmatrix} 1 & 2 \\ 0 & -31 \end{bmatrix}$$

S is a basis for $\mathbb{R}^2$, as the REF of the matrix whose cols. are the vectors in S has a pivot in ea. row & ea. col.

Consider $P_2 := \{ p(t) = at^2 + bt + c \ : \ a, b, c \in \mathbb{R} \}$.

Find a basis for P_2.

i.e., Find a linearly independent subset of P_2 that spans P_2.

Try: $\boxed{\{ 3t^2 + 4t, \ t^2 + t, \ 1 \}}$.

① Linear independence?

$\overset{?}{\exists}$ $c_1, c_2, c_3 \in \mathbb{R}$, not all zero, s.t.

$$c_1(3t^2 + 4t) + c_2(t^2 + t) + c_3(1) = 0 \ ?$$

$$(3c_1 + c_2)t^2 + (4c_1 + c_2)t + c_3 = 0$$

$$\left.\begin{aligned} 3c_1 + c_2 &= 0 \\ 4c_1 + c_2 &= 0 \\ c_3 &= 0 \end{aligned}\right\}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \begin{array}{c} \tfrac{1}{3}R1 \\ R2-\tfrac{4}{3}R1 \\ R3 \end{array} \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A pivot in every col., so the only c_1, c_2, c_3 that make

$$c_1(3t^2+4t) + c_2(t^2+t) + c_3(1) = 0$$

are $c_1 = c_2 = c_3 = 0$.

So $\underset{S:=}{\{}3t^2+4t,\ t^2+t,\ 1\}$ is lin. indep.

(2) Does S span P_2 ?

i.e., Does $at^2+bt+c = c_1(3t^2+4t) + c_2(t^2+t) + c_3(1)$

have a sol'n $\langle c_1, c_2, c_3\rangle$ for each $a, b, \& c \in \mathbb{R}$?

$$at^2+bt+c = c_1(3t^2+4t) + c_2(t^2+t) + c_3(1)$$

$$at^2+bt+c = (3c_1+c_2)t^2 + (4c_1+c_2)t + c_3$$

$$3c_1 + c_2 = a$$
$$4c_1 + c_2 = b$$
$$c_3 = c$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & a \\ 4 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right] \sim \begin{array}{c} \tfrac{1}{3}R1 \\ R2 - \tfrac{4}{3}R1 \\ R3 \end{array} \underbrace{\left[\begin{array}{ccc|c} 1 & 1/3 & 0 & a/3 \\ 0 & -1/3 & 0 & b - 4a/3 \\ 0 & 0 & 1 & c \end{array}\right]}_{\text{REF}} \sim$$

$$\sim \begin{array}{c} R1 + R2 \\ -3R2 \\ R3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & b-a \\ 0 & 1 & 0 & -3b+4a \\ 0 & 0 & 1 & c \end{array}\right].$$

So:
$$c_1 = b - a$$
$$c_2 = -3b + 4a$$
$$c_3 = c$$

makes

$$\boxed{\boxed{at^2 + bt + c} = c_1(3t^2 + 4t) + c_2(t^2 + t) + c_3(1)}$$

$$at^2 + bt + c \stackrel{?}{=} (b-a)(3t^2 + 4t) + (-3b+4a)(t^2+t) + c$$

$$= [3(b-a) + (-3b+4a)]t^2 + [4(b-a) + (-3b+4a)]t + c$$

$$= at^2 + bt + c \quad \underline{\underline{\checkmark}}.$$

So S spans P_2.

Therefore, S is a basis for P_2.

Thm. If S is a basis for a v.s. V, and T is a basis for V, then $|S| = |T|$.

Def'n. For a vector space V, the dimension of V is the # of vectors in any basis of V.

Denoted $\dim(V)$.

Ex. . $\dim(\mathbb{R}^2) = 2$.

. $\dim(P_2) = 3$.

Let $V := \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. $V \subseteq \mathbb{R}^3$. Notice:

V is a vector space, since ① V is closed under add'n, and ② V is closed under scalar multiplicat'n.

A basis for V is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

① Linear indep.: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

There's a pivot in every column.

(2) Assume $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \in V$. Can we find $c_1, c_2 \in \mathbb{R}$

s.t. $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$. So $a = c_1$, $b = c_2$ does the

job, and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans V.

(It doesn't span $\mathbb{R}^3$, but that's okay — we were asked t/prove tht. S was a basis for V, not for $\mathbb{R}^3$.)

YES, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for V.

So $\dim(V) = 2$.

L18, ct'd.

- The set $\{\vec{0}\}$ has dimension 0 (by convention).

Say you have a vector space V and a subset $S \subseteq V$.

S spans V

S is lin. indep.

S has n many vectors, $n = \dim(V)$