L19: April 6, 2017.

Last time: Bases / Lin. Indep. / Span

This time: Linear transformations

kernel / Rank / Nullity / Range

Remember: $T: V \to W$ is a linear transformation if:

$\forall v \in V, \forall w \in V, \forall c \in \mathbb{R}$:

(1) $T(v+w) = T(v) + T(w)$

(2) $T(cv) = c \cdot T(v)$

Def. A l.t. $L: V \to W$ is one-to-one if

$\forall v, w \in V$, $\begin{cases} v \neq w \Rightarrow L(v) \neq L(w). \\ L(v) = L(w) \Rightarrow v = w. \end{cases}$

Example. $L:\mathbb{R}^2\to\mathbb{R}^2$, $L\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x+y\\x-y\end{bmatrix}$

Check if one-to-one:

Assume that $v_1:=\begin{bmatrix}a_1\\a_2\end{bmatrix}$, $v_2:=\begin{bmatrix}b_1\\b_2\end{bmatrix}$

Suppose $\boxed{L(v_1)=L(v_2)}$, i.e.,

$$\begin{bmatrix}a_1+a_2\\a_1-a_2\end{bmatrix}=\begin{bmatrix}b_1+b_2\\b_1-b_2\end{bmatrix}, \quad \text{i.e.,}$$

$$① \quad a_1+a_2=b_1+b_2 \quad \text{and}$$
$$② \quad a_1-a_2=b_1-b_2.$$

To ①, add the quantity a_1-a_2 to both sides:

$$(a_1+a_2)+(a_1-a_2)=(b_1+b_2)+(a_1-a_2)$$

$$2a_1=(b_1+b_2)+(a_1-a_2)$$

Use ② to substitute $a_1-a_2=b_1-b_2$:

$$2a_1=(b_1+b_2)+(b_1-b_2)$$

$$2a_1=2b_1$$

$$\boxed{a_1=b_1.}$$

Substitute: $a_1+a_2=\overset{=b_1}{a_1}+b_2 \iff \boxed{a_2=b_2}.$

We've found that $v_1 = v_2$. Yes, L is 1-1.

T/prove a l.t. is one-to-one, take two arbitrary vectors in the domain, and assume L maps them to the same elt. of the range. Then prove that this assumption forces the two original vectors to be equal.

Example. Let $L: \mathbb{R}^3 \to \mathbb{R}^2$, s.t. $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$.

Is this one-to-one?

Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, let $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 52 \end{bmatrix}$

$L\left(\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\qquad$ $L\left(\begin{bmatrix} 3 \\ 1 \\ 52 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Def. Let $L: V \to W$ be a l.t. The kernel of L, denoted $\ker(L)$ is the subset of V consisting of all vectors $\vec{v}$ s.t. $L(\vec{v}) = \vec{0}_W$.

"All of the things from V tht. map to zero in W".

If L is one-to-one, then $\ker(L) = \{\vec{0}_v\}$.

Why? — why does the zero vector in V always map to the zero vector in W? under a l.t.

— why, if $L: V \to W$ is linear, does
$$L(\vec{0}_V) = L(\vec{0}_W) \quad ?$$

Let $\vec{v} \in V$.

$L(\vec{0}_V) = L(0 \cdot \vec{v})$, because $0 \cdot \vec{v} = \vec{0}_V \quad \forall \vec{v} \in V$.
from v.s. axioms

v.s. axiom: $\exists! \vec{0}_V \in V$ s.t. $\forall \vec{v} \in V$,
$$\vec{0}_V + \vec{v} = \vec{v}.$$

$\vec{0}_V$ is unique, as we proved (by assuming two zero vectors $\vec{0}_1$ and $\vec{0}_2$, and showing $\vec{0}_1 + \vec{0}_2 = \vec{0}_2$ and $\vec{0}_1 + \vec{0}_2 \underset{\text{v.s. axiom}}{=} \vec{0}_2 + \vec{0}_1 = \vec{0}_1$ since $\vec{0}_1$ was a zero vector. So $\vec{0}_1 = \vec{0}_2$.).

Why is $0 \cdot \vec{v} = \vec{0}_V$? ✓

$$0\cdot\vec{v} = (0+0)\vec{v}$$

$$\boxed{0\cdot\vec{v}} = 0\cdot\vec{v} + \boxed{0\cdot\vec{v}}$$ (v.s. axiom: $(c+d)\odot\vec{v} = c\odot\vec{v} \oplus d\odot\vec{v}$)

v.s. axiom: $\forall\, \vec{u}\in V,\ \exists\, -\vec{u}\in V$ s.t. $\vec{u} + (-\vec{u}) = \vec{0}_V$

Let $-(0\cdot\vec{v})$ be the additive inv. of $0\cdot\vec{v}$.

Then $0\cdot\vec{v} = 0\cdot\vec{v} + 0\cdot\vec{v}$

$$\underbrace{-(0\cdot\vec{v}) + 0\cdot\vec{v}}_{\vec{0}_V} = 0\cdot\vec{v} + 0\cdot\vec{v} + -(0\cdot\vec{v})$$

$$= 0\cdot\vec{v} + \underbrace{[0\cdot\vec{v} + -(0\cdot\vec{v})]}_{\vec{0}_V}$$

$$\vec{0}_V = \underbrace{0\cdot\vec{v} + \vec{0}_V}_{0\cdot\vec{v}}$$

$L: V\to W$

$$\vec{0}_V = 0\cdot\vec{v}$$

Fix $\vec{v}\in V$.

$$L(\vec{0}_V) = L(0\cdot\vec{v})$$

$= 0\cdot L(\vec{v})$ because $0\in\mathbb{R}$, $\vec{v}\in V$, and L was linear.

$= \vec{0}_W$, because $L(\vec{v})\in W$ and 0 is the zero scalar.

So, $\vec{0}_V \in \ker(L)$, when $L: V\to W$ is linear.

The kernel of a l.t. is always nonempty.

EX. $L : \mathbb{R}^3 \to \mathbb{R}^2$, $\quad L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$.

What is $\ker(L)$?

Suppose $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

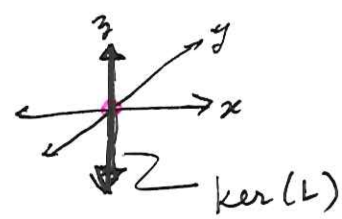
By def'n of L, $\quad L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$.

So if $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$.

$$\ker(L) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, x, y, z \in \mathbb{R} : x = y = 0 \right\}$$

or $\ker(L) = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, z \in \mathbb{R} \right\}$.



Ex. $L:\mathbb{R}^2\to\mathbb{R}^2$, $L\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x+y\\x-y\end{bmatrix}$.

$\ker(L)=\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$, because we showed L was one-to-one.

Ex. $L:\mathbb{R}^4\to\mathbb{R}^2$ $L\left(\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\right)=\begin{bmatrix}x+y\\z+w\end{bmatrix}$.

Not one-to-one: $L\left(\underbrace{\begin{bmatrix}2\\-2\\1\\-1\end{bmatrix}}_{\neq}\right)=\begin{bmatrix}2-2\\1-1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$$\|$$

$L\left(\begin{bmatrix}1\\-1\\2\\-2\end{bmatrix}\right)=\begin{bmatrix}1-1\\2-2\end{bmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$. ✓

$\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\in\ker(L)$ if $L\left(\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}$,

i.e., if $\begin{bmatrix}x+y\\z+w\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$,

i.e., if $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$ solves $\begin{cases}x+y=0\\z+w=0\end{cases}$,

i.e., if $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$ solves $\underbrace{\begin{array}{c}\begin{array}{cccc}x & y & z & w\\ \downarrow & \downarrow & \downarrow & \downarrow\end{array}\\ \begin{bmatrix}1 & 1 & 0 & 0\\0 & 0 & 1 & 1\end{bmatrix}\end{array}}_{\text{RREF}}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

i.e., if $x = -y$ and $z = -w$,

i.e., if
$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = y \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad y, w \in \mathbb{R}.$$

So $\ker(L)$ ~~contains~~ consists of all <u>linear combinations</u> of

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

$$\ker(L) = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\} \right).$$

Is $\ker(L)$ a subspace of $\mathbb{R}^4$?

(1) Add'n?

S $\vec{v} \in \ker(L)$, $\vec{w} \in \ker(L)$. So $\exists\, c_1, c_2$ s.t.

$$\vec{v} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$ and $\exists\, d_1, d_2$ s.t.

$$\vec{w} = d_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Then
$$\vec{v} + \vec{w} = \left(c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right) + \left(d_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

v.s. axioms (or Euclidean $\mathbb{R}^4$ rules)

$$= (c_1 + d_1) \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + (c_2 + d_2) \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\in \ker(L)$$

(2) Mult.:

If $\vec{v} \in \ker(L)$, so $\exists\, c_1, c_2 \in \mathbb{R}$ s.t.

$$\vec{v} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

and if $c \in \mathbb{R}$.

Then $$c\vec{v} = c\left(c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

$$c\vec{v} = (c \cdot c_1) \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + (c \cdot c_2) \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$c\vec{v} \in \ker(L)$.

So, $\ker(L)$ is a vector subspace of $\mathbb{R}^4$.

EXAM

6

$P = \{$ all polynomials of any degree $\}$

Show P is infinite-dimensional.

Hint: Suppose that $B \subseteq P$ is a finite basis for P.

Say, $B \doteq \{p_1(t), p_2(t), \dots, p_n(t)\}$. Let d_j be the degree of $p_j(t)$, $j \in [1, n] \cap \mathbb{N}$.

~~$p_2 = t^{n+1} + 4$, $p_n = 7$~~

~~$d_2 = n+1$, $d_n = 0$~~

Let $m := \max_{1 \le j \le n} d_j$.

Let $p(x) \in P$ be a polynomial of degree $m+1$.
Then since B was a basis for P, B spans P, so $\exists\, c_1, \dots, c_n$ s.t.

$c_1 p_1 + \dots + c_n p_n = p(x)$.

However, taking linear combinations of $p_1, p_2, \dots, p_n$, e.g.

$$c_1 p_1(t) + c_2 p_2(t) + \dots + c_n p_n(t),$$

will yield polynomials whose degree is, at most, m.

So $c_1 p_1 + \dots + c_n p_n \neq p(x)$. ↯ # $\rightarrow\leftarrow$

If $A \in \mathbb{R}^{3 \times 5}$, then A has at most 3 pivots (bc. it has only 3 rows). However, A has 5 columns, and so there has to be at least one column of A without a pivot.

So $\{v_1, \dots, v_5\} \subseteq \mathbb{R}^3$ is linearly dependent.

$$\boxed{7 + 9 + 21 + 13 = \boxed{20 + 30} = 50} \checkmark$$

$$\cancel{\boxed{7 + 9 = 16} + 21} = 37 + 13 = 50$$

$$6 + 3 = \underline{5} + 4$$

$$6 + 3 = 9 \cancel{+4} = 13$$

$$\boxed{6 + 3 = \underline{9} + 4}$$