Last time: Kernel

This time: Rank / nullity / Range.

Def'n: If $L: V \to W$ is a l.t., then the range of L, denoted $\text{ran}(L)$, is the set of all vectors in W that are images, under L, of vectors in V.

i.e., $\text{ran}(L) = \{ \vec{w} \in W : \exists \vec{v} \in V \text{ s.t. } L(\vec{v}) = \vec{w} \}$.

Note: If $\text{range}(L) = W$, then we say that L is "onto".

Thm: If $L: V \to W$ is a l.t., then $\text{ran}(L)$ is a vector subspace of W.

Pf: To show ran(L) is a v. subspace of W, it suffices to show closure of $\text{ran}(L)$ under vector add'n, and under scalar multiplication.

$\vec{w}_1 \in \text{ran}(L)$ and $\vec{w}_2 \in \text{ran}(L)$. (Want to show: $\vec{w}_1 + \vec{w}_2 \in \text{ran}(L)$)

i.e., $\exists \vec{v}_1 \in V$ s.t. $L(\vec{v}_1) = \vec{w}_1$ and $\exists \vec{v}_2 \in V$ s.t. $L(\vec{v}_2) = \vec{w}_2$.

$$\vec{w}_1 + \vec{w}_2 = L(\vec{v}_1) + L(\vec{v}_2) = L(\vec{v}_1 + \vec{v}_2),$$

L was a linear transformation

and note that since V was a vector space, it is closed under add'n, and therefore $\vec{v}_1 + \vec{v}_2 \in V$, as $\vec{v}_1 \in V$ and $\vec{v}_2 \in V$.

Pf (ct'd)

So, $L(\vec{v}_1+\vec{v}_2) = \vec{w}_1+\vec{w}_2$, and $\vec{v}_1+\vec{v}_2 \in V$, so $\vec{w}_1+\vec{w}_2 \in \operatorname{ran}(L)$.

Suppose $\vec{w} \in \operatorname{ran}(L)$, and $\vec{v} \in V$ is s.t. $L(\vec{v}) = \vec{w}$.

Suppose $c \in \mathbb{R}$. (Want t/show: $c\vec{w} \in \operatorname{ran}(L)$.)

$$c\vec{w} = cL(\vec{v}) \underset{\text{L was linear}}{=} L(c\vec{v}).$$

And $c\vec{v} \in V$, because V was a v.s., and is thus closed under scalar mult.

So $c\vec{w} \in \operatorname{ran}(L)$, because $L(c\vec{v}) = c\vec{w}$. □

Q: Is $\vec{0}_W \in \operatorname{ran}(L)$?

A: Yes! ("sure" – Meg)

- $L(\vec{0}_V) = \vec{0}_W$.
- Uhh... we just proved $\operatorname{ran}(L)$ is a v. subsp. of W, so it has to have a zero vector, and the zero vector is unique (we proved this earlier), so since $\vec{0}_W$ is a zero vector, it must be identical to the $\vec{0}_{\operatorname{ran}(L)}$.

Example. Let L be the l.t. $L: \mathbb{R}^3 \to \mathbb{R}^2$

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$ Is L onto?

i.e., If $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, does there have to exist $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ s.t. $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$?

.... Sure! In particular

$$L\left(\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ for } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \text{ arbitrary.}$$

The l.t. is onto.

Ex. Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}}_{=:A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

(a) Is L onto?

(b) Find a basis for $\operatorname{ran}(L)$.

(c) Find $\ker(L)$.

(d) Is L one-to-one?

(a) Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$. Question: $\overset{?}{\exists} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ s.t. $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Note: $L(\vec{x}) = A\vec{x}$.

Given $\vec{b} \in \mathbb{R}^3$, can we solve $A\vec{x} = \vec{b}$?

Observation: If A^{-1} exists, then L is onto.

Ques: Does the converse hold? i.e., Is it true that if L is onto, then A^{-1} exists?

Note: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 2 & b \\ 2 & 1 & 3 & c \end{array}\right] \sim \begin{array}{l} R1 \\ R2-R1 \\ R3-(R2+R1) \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b-a \end{array}\right]$

We know that unless $c-b-a=0$, there is no sol'n to $A\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

So L is not onto.

(b) Find a basis for ran(L).

We know: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \operatorname{ran}(L)$ only when $c = b + a$.

$$\operatorname{ran}(L) = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \text{ s.t. } c = b + a \right\}.$$

$$\operatorname{ran}(L) = \{ \vec{b} \text{ s.t. } \exists \vec{x} \text{ s.t. } A\vec{x} = \vec{b} \};$$

note: the set of all images of vectors in $\mathbb{R}^3$ under L (where $L(\vec{x}) = A\vec{x}$) is exactly the same as the set of all linear combinations of the columns of A.

<u>recall</u>: $\begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$

... this is the same as the column space col(A) !!

So, a basis for col(A) is also a basis for ran(L), since $L(\vec{x}) = A\vec{x}$ and $\operatorname{col}(A) = \operatorname{ran}(L)$.

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$ so the pivots are in cols 1 & 2;

we therefore take our basis to be cols 1 & 2 of A itself:

a basis for ran(L) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

L20, ct'd. 6

$$\operatorname{ran}(L) = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ s.t. } c = b + a \right\}$$

a basis for ran(L) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. $\dim(\operatorname{ran}(L)) = 2$.

Test: $a\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ a+b \\ a+a+b \end{bmatrix}$ ✓ (B spans ran(L))

$a\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, i.e., $\begin{bmatrix} a \\ a+b \\ 2a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

So $a = 0$, and $\therefore \begin{bmatrix} 0 \\ b \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $b = 0$. ✓ (B is lin. indep.)

(c) Find ker(L) ; $L(\vec{x}) = A\vec{x}$, $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$.

i.e., find all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ s.t. $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

i.e., Solve $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right] \sim \begin{array}{l} R1 \\ R2-R1 \\ R3-(R2+R1) \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Let $z \in \mathbb{R}$. Then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ solves $A\vec{x} = \vec{0}$.

So $\ker(L) = \left\{ a\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \text{ s.t. } a \in \mathbb{R} \right\}$.

A basis for ker(L) is therefore $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Another basis for ker(L) is $\left\{ \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \right\}$.

$$\dim(\ker(L)) = 1.$$

(d) Is L one-to-one?

i.e., $\overset{?}{\exists}\ \vec{b} \in \mathbb{R}^3$ s.t. $\exists \vec{x} \in \mathbb{R}^3$ and $\exists \vec{y} \in \mathbb{R}^3$, $\vec{x} \neq \vec{y}$, s.t. $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$?

Recall: A l.t. $\underset{L: V \to W}{\wedge}$ is one-to-one IFF $\ker(L) = \{\vec{0}_V\}$.

$$\ker(L) = \left\{ a \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : a \in \mathbb{R} \right\} \neq \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \text{ so,}$$

no, L is not one-to-one.

For this example,

$$\begin{array}{r} \dim(\mathrm{ran}(L)) = 2 \\ +\ \dim(\ker(L)) = 1 \\ \hline \dim(\mathrm{dom}(L)) = \dim(\mathbb{R}^3) = 3. \end{array}$$

$$\boxed{\begin{array}{l} \text{THM. For a linear transf. } L: V \to W, \\ \dim(\mathrm{ran}(L)) + \dim(\ker(L)) = \dim(V) \end{array}}$$

This thm is called the <u>RANK-NULLITY</u> theorem,

because $\dim(\ker(L)) =: \cancel{\text{rank}}\ \text{nullity}(L)$, and $\dim(\cancel{\ker}\ \mathrm{ran}(L)) = \mathrm{rank}(L)$.

The rank-nullity theorem applies to matrices as well.

What we got from this example:

- $\dim(\ker(L)) = 1 \neq 0$, so L was not one-to-one.

  (if $\dim(\ker(L)) = 0$, then $\ker(L) = \{\vec{0}_3\}$, so w/b one-to-one)

- $\dim(\mathrm{ran}(L)) = 2 \neq 3$, so L wasn't onto

  (if $\dim(\mathrm{ran}(L)) = 3$, then since $\mathrm{ran}(L)$ was a subspace of $\mathbb{R}^3$ and $\dim(\mathbb{R}^3) = 3$, would have had $\mathrm{ran}(L) = \mathbb{R}^3$, so L w/b onto).

THM: $L: V \to W$, a l.t., is

- ONE-TO-ONE if $\text{nullity}(L) = 0$
- ONTO if $\text{rank}(L) = \dim(W)$.