L21: Tues., Apr. 18.

Last time: Rank - nullity
Range / onto

This time: Quick recap of... the class?
Determinants.

Let $A \in \mathbb{R}^{n \times n}$. The following stmts are equivalent:

- A rowreduces to I_n (A is row-equiv. to I_n)
- For any $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique sol'n

  $\boxed{\vec{x} = \langle x_1, x_2, \dots, x_n \rangle}$

  $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n \end{bmatrix}$
- $A\vec{x} = \vec{0}$ has only the trivial sol'n
- $\dim(\mathrm{col}(A)) = n$. $\mathrm{col}(A) = \mathbb{R}^n$

  $A\vec{x} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n$
- $\dim(\mathrm{nul}(A)) = 0$. $\mathrm{nul}(A) = \{\vec{0}_n\}$
- A^{-1} exists ($A \in \mathbb{R}^{n \times n}$ implies that., further $A^{-1}_L = A^{-1}_R = A^{-1}$)
- The columns of A are linearly independent
- The rows of A are linearly independent.
- The columns of A span $\mathbb{R}^n$
- The rows of A span $\mathbb{R}^n$.
- $\underbrace{\text{The l.t. } L : \mathbb{R}^n \to \mathbb{R}^n \text{ s.t. } L(\vec{x}) = A\vec{x},}_{\mathrm{ran}(L) = \mathrm{col}(A)}$ has range $\underline{\mathbb{R}^n}$.
- The l.t. above has kernel $\ker(L) = \mathrm{nul}(A) = \{\vec{0}_n\}$

L21, ct'd.

If $|S| = n$, then $\exists$ $n!$ many permutatio of S.

A permutatio $j_1 j_2 \cdots j_n$ of $S := \{1, \ldots, n\}$ is said to have an <u>inversion</u> if a larger integer j_r precedes a smaller integer j_s.

EX. 4213 has an inversion

3241 — " — : $3>2$; $3>1$; $2>1$; $4>1$: 4 inv.

$\boxed{2134}$ — " — : one inversion

$\boxed{1234}$ DOESN'T HAVE AN INVERSION

$\boxed{4321}$ has "all the inversions"

ex.
$$\boxed{\begin{array}{c} 4>3 \\ 4>2 \\ 4>1 \\ 3>2 \\ 3>1 \\ 2>1 \end{array}}$$
6 inv.

~~ex. $4>3$ $3>2$ $2>1$~~

p. 171 of textbook :

Every permutatio can be built by successively swapping pairs of objects.

EX. $1234 \to 2134 \to 2314 \to 2341 \to 3241 \to 3421 \to 4321$

(6 swaps)

EX. $1234 \to 2134$ (1 swap)

- The l.t. is onto, because $\operatorname{ran}(L) = \mathbb{R}^m$.
- The l.t. is one-to-one because $\operatorname{nul}(A) = \ker(L) = \{\vec{0}_n\}$.

Determinants.

Recall (?): Let $S := \{1, 2, \dots, n\}$ be the set of the 1st n many positive integers. An arrangement $j_1 j_2 j_3 \cdots j_n$ of the elts. of S is called a permutation of S.

Ex. If $S = \{1, 2, 3, 4\}$, then

4213 is a perm. of S
3241 — " —
2134 — " —
1234 — " —
4321 — " —

There are $\underbrace{\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}}_{4!}$ many ~~choices~~ permutations of S.

$4!$ "four factorial"

Ex. The permutation 4132 of S corresponds to a function $f : S \to S$ defined by

$$f(1) = 4$$
$$f(2) = 1$$
$$f(3) = 3$$
$$f(4) = 2.$$

Ex. 1234 → 2134 → 2314 → 3214 → 3241

(4 swaps)

IDEA: The # of inversions of a permutat'n is equal to the # of swaps necessary to get that permutat'n from 123... n.

("Proof by examples" - see above)

(Actual proof - see p. 170-171 in text)

- If the # of swaps required t/build a permutat'n is EVEN, then we call that permutat'n an EVEN PERMUTATION
- ——— " ——————————— ODD, —— " ——— ODD —"—.

For $\underbrace{\text{any set } S}_{\text{except } s=\{1\}}$, exactly half of the permutat'ns are odd; the other —— " ——— are even.

e.g., $S=\{1,2\}$ has $|S|=2$, and $2!$ is even

$S=\{1,2,\ldots,n\}$ has $|S|=n$, and $n! = \left[n\cdot(n-1)\cdot\ldots\,4\cdot 3\cdot 1\right]\cdot 2$