Def'n. The sign function sends permutations to an elt. of $\{-1, 1\}$, with rule:

$$\operatorname{sgn}(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ is even} \\ -1, & \text{if } \sigma \text{ is odd} \end{cases}$$

Def'n. The <u>determinant</u> of an $n \times n$ matrix A is:

$$\det(A) := \sum_{\sigma} \operatorname{sign}(\sigma)\, a_{1\sigma(1)}\, a_{2\sigma(2)}\, a_{3\sigma(3)} \cdots a_{n\sigma(n)}.$$

Notes:

- The sum is over all permutat'ns of n elts.: i.e., over all elts. of $\{\sigma : \sigma \text{ maps } \{1,\dots,n\} \text{ to } \{1,\dots,n\}\}$.
- Each summand consists of the product of n many entries of A, times the sign of the permutat'n corresponding to that summand.
- In any given summand, each factor in the product that comprises that summand comes from a different row of the matrix A.

Thm. If A contains a row of zeros, then $\det(A) = 0$.

Pf. $\{a_{ij} = 0$, for i fixed and for all $j \in \{1,\dots,n\}\}$. Then $\forall \sigma$, $a_{i\sigma(i)} = 0$.
So each summand is zero; therefore, $\det(A) = 0$. $\square$

Example. Recall that the set $\{1, \dots, n\}$ has $\underline{n!}$ permutations. There are very many summands in the general formula for a determinant.

2×2 case: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

The permutations of $\{1,2\}$ are 12 and 21.

So
$$\det(A) = \sum_{\sigma \in \{12, 21\}} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)}$$
$$= \operatorname{sgn}(12)\, a_{11} a_{22} + \operatorname{sgn}(21)\, a_{12} a_{21}$$
$$\det(A) = a_{11} a_{22} - a_{12} a_{21}$$

Ex. $\det \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix} = 4(0) - 1(2) = 0 - 2 = -2.$

Ex. $\det \begin{pmatrix} 4 & b \\ 2 & 0 \end{pmatrix} = 4(0) - b(2) = -2b$

L82, ct'd.

EX. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

The permutat'ns of $\{1,2,3\}$ are
σ_1: 1 2 3 : $sgn(\sigma_1) = 1$
σ_2: 2 1 3 : $sgn(\sigma_2) = -1$
σ_3: 3 2 1 : $sgn(\sigma_3) = -1$
σ_4: 3 1 2 : $sgn(\sigma_4) = 1$
σ_5: 2 3 1 : $sgn(\sigma_5) = 1$
σ_6: 1 3 2 : $sgn(\sigma_6) = -1$

$$\det(A) = \sum_{i=1}^{6} sgn(\sigma_i)\, a_{1\sigma_i(1)} a_{2\sigma_i(2)} a_{3\sigma_i(3)}$$

$$= sgn(\sigma_1)\, a_{11} a_{22} a_{33} + sgn(\sigma_2)\, a_{12} a_{21} a_{33} + sgn(\sigma_3)\, a_{13} a_{22} a_{31}$$
$$+ sgn(\sigma_4)\, a_{13} a_{21} a_{32} + sgn(\sigma_5)\, a_{12} a_{23} a_{31} + sgn(\sigma_6)\, a_{11} a_{23} a_{32}.$$

$$\boxed{\det(A) = a_{11} a_{22} a_{33} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31} + a_{13} a_{21} a_{32} + a_{12} a_{23} a_{31} - a_{11} a_{23} a_{32}}$$

Ex. $A = \begin{pmatrix} 1 & 0 & 7 \\ 2 & 4 & 5 \\ 4 & 0 & 0 \end{pmatrix}$

$$\det(A) = 1\cdot 4\cdot 0 - 0\cdot 2\cdot 0 - 7\cdot 4\cdot 4 + 7\cdot 2\cdot 0 + 0\cdot 5\cdot 4 - 1\cdot 5\cdot 0$$

$$= 0 - 0 - 112 + 0 + 0 - 0$$

$$= -112$$

~~det(A)~~

Recall: If $A = \begin{pmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ \overset{-}{a_{21}} & \overset{+}{a_{22}} & \overset{-}{a_{23}} \\ \overset{+}{a_{31}} & \overset{-}{a_{32}} & \overset{+}{a_{33}} \end{pmatrix}$, then

Cofactor expansion

$$\det(A) = a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$\boxed{\det(A) = \underline{a_{11}a_{22}a_{33}} - \underline{a_{11}a_{32}a_{23}} - \underline{a_{12}a_{21}a_{33}} + \underline{a_{12}a_{31}a_{23}} + \underline{a_{13}a_{21}a_{32}} - \underline{a_{13}a_{31}a_{22}}}$$

$$\det(A) = -a_{21}\det\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33}\end{pmatrix} + a_{22}\det\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}\end{pmatrix} -$$
$$- a_{23}\det\begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32}\end{pmatrix}$$

For any fixed $i \in \{1, \dots, n\}$,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det\left(\begin{matrix}\text{the submatrix of } A \text{ obtained} \\ \text{by taking all entries not in} \\ \text{the same row or col. as } a_{ij}\end{matrix}\right).$$

$A \in \mathbb{R}^{n\times n}$

Also, for any fixed $j \in \{1, \dots, n\}$,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det\Big(\qquad \Big).$$

Example.

$$A = \begin{pmatrix} 1 & 0 & 5 & 8 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ 4 & 0 & 0 & 0 \end{pmatrix}$$

$\det(A) =: |A|$

$$\det(A) = -4\det\begin{pmatrix} 0 & 5 & 8 \\ 2 & 1 & 3 \\ 2 & 1 & 1\end{pmatrix} = -4\left[-5\begin{vmatrix} 2 & 3 \\ 2 & 1\end{vmatrix} + 8\begin{vmatrix} 2 & 1 \\ 2 & 1\end{vmatrix}\right]$$
$$= -4\left[-5(2\cdot 1 - 2\cdot 3) + \underbrace{8(2\cdot 1 - 2\cdot 1)}_{0}\right]$$
$$= -4(-5)(-4) = -80$$