L23: Thurs., Apr. 27 (last class)

Determinants, rules for determinants.

- The determinant of a diagonal matrix is the product of its diagonal entries.

$$\det M = \sum_{\sigma} \operatorname{sgn}(\sigma)\, m_{1\sigma(1)}\, m_{2\sigma(2)}\, m_{3\sigma(3)} \cdots m_{n\sigma(n)}$$

for a diagonal matrix, $m_{ij} = 0$ if $i \neq j$, so only one permutation can "survive" in the sum — the "identity" permutation with $\sigma(i) = i$, $\forall i \in [1, n] \cap \mathbb{N}$.

$$\det M = m_{11}\, m_{22}\, m_{33} \cdots m_{nn}, \text{ for diagonal matrices } M.$$

Q: What's the determinant of I_n?

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \qquad \det(I_n) = 1.$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1.$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \cdot \underbrace{\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = 1 \cdot (1)$$

$$= 1.$$

What happens to det M when swapping rows of M?

For σ a permutation of $\{1, \dots, n\}$, let $\hat{\sigma}$ be the permutation obtained by swapping positions i and j.

Then $\operatorname{sgn}(\hat{\sigma}) = -\operatorname{sgn}(\sigma)$.

Let $\hat{M}$ be the matrix obtained by swapping rows i and j of M. (Assume, WLOG, $i<j$.)

$$\det \hat{M} = \sum_{\sigma} \operatorname{sgn}(\sigma)\, m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{i\sigma(i)} \cdots m_{j\sigma(j)} \cdots m_{n\sigma(n)}$$

$$= \sum_{\sigma} \operatorname{sgn}(\sigma)\, m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{i\sigma(j)} \cdots m_{j\sigma(i)} \cdots m_{n\sigma(n)}$$

$$= \sum_{\sigma} (-\operatorname{sgn}(\hat{\sigma}))\, m_{1\hat{\sigma}(1)} m_{2\hat{\sigma}(2)} \cdots m_{i\hat{\sigma}(i)} \cdots m_{j\hat{\sigma}(j)} \cdots m_{n\hat{\sigma}(n)}$$

(The only diff. btwn. σ and $\hat{\sigma}$ is that $\hat{\sigma}(i) = \sigma(j)$ and $\hat{\sigma}(j) = \sigma(i)$. Otherwise, $\hat{\sigma}(k) = \sigma(k)$ for $k \in [1, n] \cap \mathbb{N} / \{i, j\}$)

$$= -\sum_{\hat{\sigma}} \operatorname{sgn}(\hat{\sigma})\, m_{1\hat{\sigma}(1)} \cdots m_{n\hat{\sigma}(n)}$$

$$= \underbrace{-\sum_{\sigma} \operatorname{sgn}(\sigma)\, m_{1\sigma(1)} \cdots m_{n\sigma(n)}}_{\det(M)}$$

$$\boxed{\det \hat{M} = -\det(M).}$$

So if M and $\hat{M}$ differ by (only) a row swap, then

$$\det \hat{M} = -\det M.$$

Q. What's $\det \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{A}$?

$\det(A) = -\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (swap last two rows)

$\boxed{\det A = -1}$ (as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is I_3).

Q. What's $\det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$?

$\det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = -\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1.$

Q. Let E_{ij} be the <u>elementary matrix</u> corresponding to a swap of rows i and j of ~~the~~ I_n.

$\det(E_{ij}) = -1.$

Q. What if a matrix has two identical rows?

$\Rightarrow$ $M \in \mathbb{R}^{n \times n}$ and row i of M = row j of M.

Let $\hat{M}$ be the matrix obtained by swapping rows i and j of M (note: $\hat{M} = M$).

$\det(\hat{M}) = -\det(M) \longleftarrow$ as we swapped rows. On the other hand,

$\det(\hat{M}) = \det(M) \longleftarrow$ as $\hat{M} = M$

so $\det(M) = -\det(M)$.

$\boxed{\text{So, } \det(M) = 0, \text{ if } M \text{ has two identical rows.}}$

Scalar multiples of rows:

Let $M := \begin{bmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — r_m — \end{bmatrix}$, $r_i \in \mathbb{R}^m$.

Let $R_i(\lambda)$ be the identity matrix with i^{th} diag. entry equal to λ (not 1), i.e.,

$$R_i(\lambda) = \begin{array}{cc} \begin{array}{ccccc} & & i^{th} \text{ col} & & \end{array} & \\ \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \lambda & & \vdots \\ & & & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} & \begin{array}{c} \\ \\ i^{th} \text{ row} \\ \\ \\ \end{array} \end{array}$$

Then, let $\hat{M} := R_i(\lambda) M = \begin{bmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — \lambda r_i — \\ \vdots \\ — r_m — \end{bmatrix}$, so

$\hat{M}$ is M with i^{th} row scaled by λ.

$$\det(\hat{M}) = \sum_{\sigma} \operatorname{sgn}(\sigma)\, m_{1\sigma(1)}\, m_{2\sigma(2)} \cdots \lambda m_{i\sigma(i)} \cdots m_{n\sigma(n)}$$

$$= \lambda \underbrace{\sum_{\sigma} \operatorname{sgn}(\sigma)\, m_{1\sigma(1)}\, m_{2\sigma(2)} \cdots m_{n\sigma(n)}}$$

- $\boxed{\det(\hat{M}) = \lambda \det(M), \text{ if } \hat{M} \text{ is } M \text{ with one row scaled by } \lambda.}$

Recall: To add $\mu \cdot r_j$ to r_i and store in r_i,

multiply $M = \begin{bmatrix} - r_1 - \\ - r_2 - \\ \vdots \\ - r_m - \end{bmatrix}$ by $S_{ij}(\mu)$, where

$S_{ij}(\mu)$ is I_m with an add'l entry: μ in the (i,j) pos'n.

e.g., if $M \in \mathbb{R}^{3\times 3}$: $M = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$, then

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M \end{pmatrix} = \begin{matrix} R1 \\ R2 + 3R1 \\ R3 \end{matrix} \begin{pmatrix} M \end{pmatrix}.$$

Let $\hat{M} := S_{ij}(\mu) M$, where $M = \begin{bmatrix} - r_1 - \\ \vdots \\ - r_m - \end{bmatrix}$, $r_i \in \mathbb{R}^m$, $i \in \{1, \ldots,$

$$\det(\hat{M}) = \sum_\sigma \operatorname{sgn}(\sigma)\, m_{1\sigma(1)} m_{2\sigma(2)} \cdots (m_{i\sigma(i)} + \mu\, m_{j\sigma(i)}) \cdots m_{n\sigma(n)}$$

$$= \underbrace{\left[\sum_\sigma \operatorname{sgn}(\sigma)\, m_{1\sigma(1)} \cdots m_{n\sigma(n)} \right]}_{\det(M)} + \underbrace{\left[\sum_\sigma \operatorname{sgn}(\sigma)\, m_{1\sigma(1)} \cdots \mu\, m_{j\sigma(i)} \cdots m_{n\sigma(n)} \right]}$$

let $\hat{\hat{M}}$ be the matrix that is M, with rows i and j identical

i.e., $\det(\hat{\hat{M}}) = 0$.

$\boxed{\begin{array}{l} \det(\hat{M}) = \det(M) \\ \text{if } \hat{M} \text{ is obtained from } M \text{ by adding scalar mult. of rows.} \end{array}}$

① Swapping rows: $\det \hat{M} = -\det M$

② Scaling a row by λ : $\det \hat{M} = \lambda \det M$

③ Taking linear combinations of rows: $\det(\hat{M}) = \det M$.

Example. Let $M := \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 2 & 2 & 2 \end{pmatrix}$.

Find det(M) by row-reducing ...

$$\underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 2 & 2 & 2 \end{pmatrix}}_{M} \sim \begin{array}{c} \frac{1}{2}R_1 \\ \boxed{\begin{array}{c} R2-2R1 \\ R3-R1 \end{array}} \end{array} \underbrace{\begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}}_{M_1} \sim \begin{array}{c} R1 \\ R2 \\ \boxed{R3-R2} \end{array} \underbrace{\begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & +1 \end{pmatrix}}_{M_2} \sim$$

$\det(M) = \frac{1}{2}\det(M_1)$

$\det(M_1) = \det(M_2)$

so $\det(M) = \frac{1}{2}\det(M_2)$

$$\begin{array}{c} R1 \\ R2-R3 \\ R3 \end{array} \underbrace{\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{M_3} \sim \begin{array}{c} \boxed{R1-\frac{1}{2}R2} \\ R2 \\ R3 \end{array} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{M_4}$$

$\det(M_2) = \det(M_3)$

so $\det(M) = \frac{1}{2}\det(M_3)$

$\det(M_3) = \det(M_4)$

so $\det(M) = \frac{1}{2}\det(M_4)$

and $M_4 = I_3$, so $\det(M_4) = \det(I_3) = 1$, so

$\det(M) = \frac{1}{2} \cdot 1 = \frac{1}{2}$. Could do a reality check and compute $\det(M)$ by cofactor expansion +/see if it's still $\frac{1}{2}$.

THM. M is invertible IFF $\det M = 0$.

Note: Either $\underbrace{\text{rref}(M) = I_n}_{\det(M) = \underset{\neq 0}{c} \cdot 1}$ or $\underbrace{\text{rref}(M) \text{ has a row of 0's.}}_{\det(M) = c \cdot 0}$

TH'M. If M and N are both ~~invertible~~ $n \times n$,

then $\det(MN) = \det(M) \cdot \det(N)$