Housekeeping.

- WeBWorK due 11:59 p.m. tonight

- Written homework due in class on Tuesday

Last time: row reduction practice.

questions?

This time:
- Linear systems $\Leftrightarrow$ vector eq'ns
  $\Leftrightarrow$ matrix eq'ns

- ~~The nature of solution sets~~
- "Formalizing" row reduction — you'll be able to program a computer to row-reduce for you!

Consider the vector equation in $\mathbb{R}^2$:

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \end{bmatrix},$$

where x_1 & x_2 are unknown scalars.

"Simplify" the LHS:

$$\begin{bmatrix} x_1 + x_2 \\ 2x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \end{bmatrix}.$$

Now... two vectors in $\mathbb{R}^2$ are equal only if their respective components are equal !

$$\left(\text{i.e.,} \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \text{ only when } a = c \text{ and } b = d. \right)$$

So this "vector equation" boils down to...

$$\begin{cases} x_1 + x_2 = 27 \\ 2x_1 - x_2 = 0 \end{cases}.$$

A LINEAR SYSTEM.

... and, we know how to solve this linear system by converting to an augmented matrix & row-reducing:

$$\begin{cases} x_1 + x_2 = 27 \\ 2x_1 - x_2 = 0 \end{cases}$$

pivot

$$\begin{bmatrix} \boxed{1} & 1 & | & 27 \\ 2 & -1 & | & 0 \end{bmatrix} \longrightarrow$$

REF

$$\begin{matrix} R1 \\ R2-2R1 \end{matrix} \begin{bmatrix} 1 & 1 & | & 27 \\ 0 & \boxed{-3} & | & -54 \end{bmatrix}$$

pivot

$$\longrightarrow \begin{matrix} R1 \\ -\frac{1}{3}R2 \end{matrix} \begin{bmatrix} 1 & 1 & | & 27 \\ 0 & 1 & | & 18 \end{bmatrix}$$

$$\boxed{\begin{matrix} x_1 = 9 \\ x_2 = 18 \end{matrix}}$$

$$\longrightarrow \begin{matrix} R1-R2 \\ R2 \end{matrix} \begin{bmatrix} 1 & 0 & | & 9 \\ 0 & 1 & | & 18 \end{bmatrix} \Big\} \text{ RREF}$$

But also... consider the following:

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} *_1 \\ *_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_1 - x_2 \end{pmatrix}$$

2×2   2×1   2×1

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

A new kind of problem... "matrix eq'ns"

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix} \quad : \text{ solve for } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

So, the questions ...

- Solve the linear system

$$\begin{cases} x_1 + x_2 = 27 \\ 2x_1 - x_2 = 0 \end{cases}$$

- Solve the vector equation

$$x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix}$$

- Solve the "matrix" equation

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix}$$

$$A\vec{x} = \vec{b}$$

... are exactly the same !

They can all be solved by row-reducing (because they are all the same problem !).

Formalizing the concept of row reduction...

On an <u>augmented matrix</u>, it is possible to perform any of the following three <u>row operations</u> ...

    (1) ROW SWAP

    (2) SCALING A ROW

    (3) TAKING THE <u>LINEAR COMBINATION</u> OF TWO ROWS
        (i.e., adding a scalar multiple of one row
           to a scalar multiple of another row)

... and end up with another augmented matrix that is <u>row-equivalent</u> to the one you started with.

QUESTIONS/EXERCISES:

    ① Every matrix is row-equivalent to itself

    ② If A is row-equivalent to B, then B is
        row-equivalent to A

    ③ If A is row-equiv. to B and B is row-equiv.
        to C, then A is row-equiv. to C.

So, ROW EQUIVALENCE is

- · REFLEXIVE
- · SYMMETRIC
- · TRANSITIVE.

Therefore, we call row equivalence an <u>equivalence relation</u>, and observe that it divides the set of all matrices into exclusive <u>equivalence classes</u>, where every~~thing~~ matrix in an equiv. class is row-equivalent to every other matrix in that class.

... So, we can perform row operations on augmented matrices and end up with ∧ matrices that are all row-equiv.
　　　　　　　　　　augmented

The goal of row reduc'n is to end up with an augmented system in row echelon form (REF) or reduced row echelon form (RREF), that's row-equivalent to the matrix you started with.

The systematic algorithm for obtaining RREF or REF from an arbitrary (augmented) ~~matrix~~ is called :

ROW REDUCTION

or

GAUSSIAN ELIMINATION

or

THE GAUSS-JORDAN METHOD :

1. Find the left-most column of the matrix that is <u>not all</u> <u>zeros</u> — it is the "PIVOT COLUMN".

$$\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 6 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

PIVOT COLUMN

2. Find the uppermost nonzero entry in the pivot column. Call it the "PIVOT".

$$\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ \boxed{2} & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

PIVOT

3. Swap rows, if necessary, so that the pivot is in the top row.

$$\begin{matrix} R3 \\ R2 \\ R1 \\ R4 \end{matrix} \begin{bmatrix} \boxed{2} & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

PIVOT

4. Scale the pivot row so that the pivot is 1.

$$\begin{matrix} \frac{1}{2}R1 \\ R2 \\ R3 \\ R4 \end{matrix} \begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

5. Add appropriate multiples of the PIVOT ROW to all other rows, so that all other entries in the pivot column are zero:

PIVOT → PIVOT COL

$$R4-2R1 \quad \begin{bmatrix} R1 & 1 & 1 & -5/2 & 1 & 2 \\ R2 & 0 & 0 & 2 & 3 & 4 \\ R3 & 0 & 2 & 3 & -4 & 1 \\ & 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

6. Repeat steps 1-5 on the submatrix that excludes the pirot row:

$$\begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{-5/2} & \cancel{1} & \cancel{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$\tilde{1}$. ① most col. not all 0's = PIVOT COL.

$$\begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

PIVOT COLUMN

$\tilde{2}$. Uppermost nonzero entry of pirot col. = PIVOT

PIVOT → PIVOT.

$\tilde{3}$. ~~Scale pivot row s.t. pirot is 1~~

Swap rows s.t. pivot is 1st in its column.

PIVOT COL.

$$\begin{bmatrix} R2 & 0 & 2 & 3 & -4 & 1 \\ R1 & 0 & 0 & 2 & 3 & 4 \\ R3 & 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$\tilde{4}$. Scale pivot row s.t. pivot value is 1.

PIVOT COL.

$$\begin{matrix} \frac{1}{2}R*1 \\ R2 \\ R3 \end{matrix} \begin{bmatrix} 0 & 1 & 3/2 & -2 & 1/2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$\tilde{5}$. Annihilate the entries in the pirot col. below the pirot

$$\begin{matrix} R1 \\ R2 \\ R3+2R1 \end{matrix} \begin{bmatrix} 0 & 1 & 3/2 & -2 & 1/2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

7. Repeat Step 6 until there are no remaining rows in the matrix.

$\tilde{6}$. Write the submatrix that excludes the top row:

$$\begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

PIVOT

PIVOT COLUMN

$\tilde{\tilde{1}}$. Identify pivot col.

$\tilde{\tilde{2}}$. — " — .

$\tilde{\tilde{3}}$. Put pivot in top of col. $\frac{1}{2}R1\begin{bmatrix} 0 & 0 & 1 & 3/2 & 2 \end{bmatrix}$

$\tilde{\tilde{4}}$. Scale pivot to 1 $R2\begin{bmatrix} 0 & 0 & 2 & 3 & 4 \end{bmatrix}$

$\tilde{\tilde{5}}$. Annihilate what's below pivot:

$$\begin{array}{c} R1 \\ R2-2R1 \end{array}\begin{bmatrix} 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\tilde{\tilde{6}}$. New submatrix is $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(7t)
8. Put the "removed" rows back:

$$\begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 1 & 3/2 & -2 & 1/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in REF.

To get RREF...

8. Find the right-most column that has a pivot. (The pivot is the **lowest** (top-down) nonzero entry in the column.)

$$\begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 1 & 3/2 & -2 & 1/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

PIVOT COL.

9. Repeat step ~~5~~ 5; annihilate all entries in pivot column **above** the pivot.

$$\begin{array}{l} R1 + \frac{5}{2}R3 \\ R2 - \frac{3}{2}R3 \\ R3 \\ R4 \end{array} \begin{bmatrix} 1 & 1 & 0 & 19/4 & 7 \\ 0 & 1 & 0 & -17/4 & -5/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$2 + \frac{5}{2}(2) = 7;$ $-2 - \frac{3}{2}\left(\frac{3}{2}\right) = -2 - \frac{9}{4}$

$1 + \frac{5}{2}\left(\frac{3}{2}\right) = 1 + \frac{15}{4} = \frac{19}{4};$ $= -\frac{8-9}{4} = -\frac{17}{4};$ $\frac{1}{2} - \frac{3}{2}(2) = \frac{1}{2} - 3$

10. Repeat steps 8-9 for the next right most column.
    ∧ pivot

$$\begin{array}{l} R1 - R2 \\ R2 \\ R3 \\ R4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 9 & 19/2 \\ 0 & 1 & 0 & -17/4 & -5/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

11. Repeat step 10 until all pivot columns have only one nonzero ~~#~~ entry, and that entry is 1.