L9: Feb. 21, 2017.

Housekeeping.

- WebWork due Thursday
- Written HW due Tuesday

Last time: Exam

Questions?

This time: Matrix inverses.

Matrix Inverses.

Example: For the algebraic eq'n $6x = 12$, we "solve for x" by multiplying both sides of the eq'n by the multiplicative inverse of 6 — that's $\frac{1}{6}$, because $6 \cdot \frac{1}{6} = 1 = \frac{1}{6} \cdot 6$.

What we'd like: to be able to solve the "matrix eq'n"

$$A\vec{x} = \vec{b}$$

by "multiplying both sides by the multiplicative inverse" — what would such an inverse look like??

- It would "undo" Gaussian elimination — record all row op'ns with elementary matrices, then take the product of those matrices.

Last time:
$$\left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\hphantom{0\;1\;1}}^{A}} & \overbrace{\hphantom{7}}^{\vec{b}} \\ 0 & 1 & 1 & 7 \\ 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right), \text{ and}$$

this was done using row op'ns represented by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1/2 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{this is the inverse of } A!}$$

So, to solve $A\vec{x} = \vec{b}$, where $A := \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 7 \\ 4 \\ 4 \end{pmatrix}$,

we multiply both sides (on the left) by $A^{-1} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$:

$$\underbrace{A^{-1}A\vec{x}} = A^{-1}\vec{b}$$

$$A^{-1}A = \begin{pmatrix} 0 & 1/2 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1}A\vec{x} = I_3\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A^{-1}\vec{b} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

$3 \times 3 \quad 3 \times 1$ ✓

So, $A^{-1}A\vec{x} = A^{-1}\vec{b} \quad \Rightarrow \quad \vec{x} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

The ~~upshot~~ up-side of solving $A\vec{x} = \vec{b}$ using A^{-1} is that it's much easier to solve again w/ different RHS.

So... how to find A^{-1}?

Could: Write an elementary matrix for each row op'n, and multiply out ... OR could record row op'ns as you row reduce.

Example. To invert the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

- Augment with the identity
- Row-reduce as usual. $\quad (A \text{ "}x\text{" } = I)$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \begin{array}{c} R2 \\ R1 \\ R3 \end{array} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\begin{array}{c} \tfrac{1}{2}R1 \\ R2 \\ R3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \tfrac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \begin{array}{c} R1 \\ R2-R3 \\ R3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \tfrac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underbrace{}_{I_3} \qquad \underbrace{}_{A^{-1}}.$$

So A^{-1} exists (RHS of augment row-reduced to I_3),

and is equal to $\begin{pmatrix} 0 & \tfrac{1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

Q: Are matrix inverses unique?

Proof: Suppose that B and C are both inverses of A, ~~and that $B \neq C$.~~

Since B is an inverse of A, $AB = I$. Also, $AC = I$.
Since equality is transitive, $AB = AC$. Multiply on the left by B: $BAB = BAC$, and since $AB = I$, we obtain $IB = IC$, i.e., $B = C$. ~~$\Downarrow \;\; \not\equiv$ Aaaaahhhh!!~~

- Matrix inverses, if they exist, are unique.
- But ! – Not all matrices are invertible.

Example. $A := \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Try to find A^{-1}.

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \begin{array}{c} \tfrac{1}{2}R2 \\ R1 \\ R3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \begin{array}{c} R1 \\ R2 \\ R3-R2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right).$$

$$\underbrace{}_{\text{RREF}}$$

This tells us that $\begin{pmatrix} 0 & 1/2 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

It also tells us that, since $\mathrm{rref}(A) \neq I$, A cannot have an inverse; i.e., A is not invertible.

In general: A matrix A is invertible

IFF

A is row-equivalent to I.

So far, we've talked only abt. LEFT INVERSES – i.e., the left inverse of A is A^{-1} if $A^{-1}A = I$.

Analogue concept: RIGHT INVERSE – the right inverse of A is A^{-1} if $AA^{-1} = I$.

Note on dimensions:

If A is not square, (i.e., if $A \in \mathbb{R}^{m \times n}$, $m \neq n$), then "I" in the above def'ns refers to different matrices!

- Ⓛ $\underset{(n \times m)}{A^{-1}} \underset{(m \times n)}{A} = \underset{n \times n}{I}$, so "$I$" means I_n.

- Ⓡ $AA^{-1} = I$, so "I" means I_m.

  $(m \times n)(n \times m) = m \times m$