Lecture 1:
Lesson and Activity Packet

MATH 330: Calculus I

September 7, 2016

How This Works: Each class, we will try to work through a packet like this (and we'll keep some local paper companies in business, I suspect; please recycle!). There are some notes, and some exercises. Individual exercises are in green boxes, and group exercises are in yellow boxes. There are also some worked examples and definitions as part of the notes. If there is ever a packet we don't complete, I'd ask you to work through the exercises and read the notes as best you can before the subsequent class; you can always ask me (Dr. Kiley) for help if you're having trouble understanding something. We will work through the packets together as a class, so please don't read too far ahead, unless you just can't help yourself...

Definition 1 (*Sequence*)

A sequence *is an ordered list of numbers*

$$a_1, a_2, \ldots, a_n, \ldots$$

The numbers in the sequence are called terms *of the sequence; for example,*

$$\underbrace{2}_{1^{st} \text{ term}}, 4, 6, \ldots, \underbrace{2n}_{n^{th} \text{ term}}, \ldots$$

When we are talking about the n^{th} term a_n in a sequence, the natural number n is called the index *of the term a_n.*

Example 1

We can describe sequences by…

- *Listing their terms explicitly,* e.g.,
  - $\{a_n\} := \{1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$
  - $\{b_n\} := \{-1, 2, -3, \ldots, n(-1)^n, \ldots\}$
- *Writing the rule in terms of a general index n (or another letter),* e.g.,
  - $\{a_n\} := \{\sqrt{n}\}_{n=1}^{\infty}$
  - $\{b_n\} := \{n(-1)^n\}_{n=1}^{\infty}$

  Sometimes when the meaning is otherwise clear, the indices are omitted from the curly brace.
- *Defining them as* functions *whose domain is the set of* natural numbers[a], e.g.,
  - $a_n := \sqrt{n}$
  - $b_n := n(-1)^n$
- *Describing them with words,* e.g.,
  - *The sequence of square roots of the natural numbers*
  - *The alternating sequence of the natural numbers*[b]

[a] The set of natural numbers is denoted $\mathbb{N}$, and is equal to $\{1, 2, 3, \ldots\}$. Note that in a way, $\mathbb{N}$ is itself a sequence that could be denoted $\{n\}_{n=1}^{\infty}$.

[b] "Alternating" is used in this way to denote the sign of each term—they switch between positive and negative.

Individual Exercise 1

List the first ten terms of each sequence:

- *The sequence of prime integers*[a].
- *The* Fibonacci sequence $\{F_n\}$, *which is defined* recursively *as follows:*

$$F_1 := 1, \quad F_2 := 1, \quad F_{n+1} := F_n + F_{n-1} \text{ for } n \geq 2.$$

- $\{3 + (-1)^n\}$

[a] An integer is *prime* if its only two divisors are 1 and itself.

Group Exercise 2

Write a formula for a general term a_n *in the following sequences:*

- $\{1, 4, 9, 16, \ldots\}$
- $\{0, 2, 0, 2, \ldots\}$
- $\left\{\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14}, \ldots\right\}$

Sometimes when we graph the terms of a sequence as dots, a long-term trend becomes clear. For example, let's compare and contrast the graphs of three sequences. The first one we'll do together:

$$\{a_n\} := \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$$

Individual Exercise 3

Graph the terms of the following two sequences:

- $\{b_n\} := \{1, \sqrt{2}, \sqrt{3}, 2, \ldots, \sqrt{n}, \ldots\}$
- $\{c_n\} := \{1, -1, 1, -1, 1, \ldots, (-1)^{n+1}, \ldots\}$

Group Exercise 4

What do you notice about the behavior of a_n, b_n, and c_n as n gets large?

For the sequences we just studied as an example, we have the following qualities:

- The terms of $\{a_n\}$ approach zero;
- The terms of $\{b_n\}$ *grow without bound*;
- The terms of $\{c_n\}$ *cycle* between -1 and 1.

Convergence is an extremely important concept in mathematics. In the above examples, we say that

- $\{a_n\}$ *converges to 0*; we write either $\lim_{n\to\infty} a_n = 0$ or $a_n \to 0$.
- $\{b_n\}$ and $\{c_n\}$ both *diverge*

Of course, sequences can converge to numbers other than zero as well. If a sequence converges, the value it converges to is called its *limit*.

To formalize the concept of convergence and limits, consider the following "*epsilon-delta*" definition[1]:

Definition 2 (*Limit of a Sequence*)

We say that the sequence $\{a_n\}$ converges to the real number L, or has the limit L, provided that the following statement is true:

> *Given any number $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$, $|a_n - L| < \varepsilon$.*

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$, and L is called the limit of the sequence.

If no such number L exists for the sequence $\{a_n\}$, then we say that $\{a_n\}$ diverges.

If you have never seen an epsilon-delta definition before, this can look intimidating. Let's illustrate what it means:

[1] There are no δ's that appear in this definition; the reason it's called an "epsilon-delta" definition is because very similar definitions of limits have been formulated for functions of a continuous spectrum of real numbers, and they require slight tweaking to deal with continuity of inputs. Don't worry too much about that if you've never seen such definitions before.

- The "statement" in the definition can be abbreviated using *quantifiers*[2], as follows:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N,\ |a_n - L| < \varepsilon.$$

- Think of Definition 2 like a game. If you claim that L is the limit of $\{a_n\}$, then no matter how small an ε someone gives you, you always have to be able to show an integer N so that past the term a_N, **all** of the subsequent terms of $\{a_n\}$ are within ε distance of L. The easiest way to show you can do this is to give a formula for N in terms of ε.

- If you claim (or are asked to show) that $a_n \to L$, then a template for the "epsilon-delta" proof of that statement follows. Use this template on your homework assignment!

  > Let $\varepsilon > 0$ be given. Then if N is any integer greater than [insert formula for N in terms of ε here], it will be true that for all $n > N$, $|a_n - L| = \cdots = \cdots < \varepsilon$. This proves that $a_n \to L$.

  You need to fill in only two parts: first, the formula for N in terms of ε, and second, you need to show that $|a_n - L| < \varepsilon$ using a string of equalities and inequalities that follow as a logical result of the formula you filled in first.

Group Exercise 5

Use your intuition (not yet an "epsilon-delta" proof) to reason what the limit of the sequence $\{\frac{1}{n}\}$ should be. We will use "epsilon-delta" reasoning to prove this statement on the next page.

[2]If you have never seen quantifiers before, do not worry. You will not be required to use them in your submitted work, but you will be required to—as mentioned in the syllabus and as you'll hear repeatedly—show your arguments using full sentences, and sometimes using quantifiers to take the place of words in mathematics can make that easier. A link will be posted on Canvas to those interested in quantifiers and other "allowable" mathematical symbols.

Example 2

To prove this assertion, we use the template, and need to figure out what to fill in as the formula for N in terms of ε. The easiest way to do that is to work backward from the desired result, using reversible (or, as we will see, sometimes just reverse) logical steps. The desired result is that $\{a_n - L\} < \varepsilon$, or $\{\frac{1}{n} - 0\} < \varepsilon$. So we start from there and work backward to find out what the restriction on n has to be.

$$\begin{aligned} \left|\frac{1}{n} - 0\right| < \varepsilon &\iff \left|\frac{1}{n}\right| < \varepsilon \\ &\Longleftarrow \frac{1}{n} < \varepsilon \\ &\iff n > \frac{1}{\varepsilon}. \end{aligned}$$

Notice that each of the steps above was either reversible (indicated by $\iff$, which stands for "if and only if" and means that the two statements on either side are logically equivalent), or was simply a backward implication (indicated by $\Longleftarrow$, which means that the statement on the left is a logical result of the statement on the right). This means that we can start from $n > \frac{1}{\varepsilon}$ and conclude that $\left|\frac{1}{n} - 0\right| < \varepsilon$.

Then we just plug the formula $N = \frac{1}{\varepsilon}$ and the arithmetic showing our logical argument (starting at $n > N$ and concluding that $\left|\frac{1}{n} - 0\right| > \varepsilon$) into our proof template:

> *Let $\varepsilon > 0$ be given. Then if N is any integer greater than $\frac{1}{\varepsilon}$, it will be true that for all $n > N$, $\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| < \left|\frac{1}{N}\right| < \varepsilon$. This proves that $\{\frac{1}{n}\} \to 0$.*

Group Exercise 6

Why is the statement $\left|\frac{1}{n}\right| < \varepsilon \Longleftarrow \frac{1}{n} < \varepsilon$ not written $\left|\frac{1}{n}\right| < \varepsilon \iff \frac{1}{n} < \varepsilon$ instead? What makes it not reversible?

Another common example of a step that is not reversible is squaring both sides of an equation where you don't know if the terms on either side might be negative.

Group Exercise 7

Prove that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Finally, we'll have our last "epsilon-delta"-type definition related to sequence convergence (or, in this case, divergence):

Definition 3

The sequence $\{a_n\}$ is said to diverge to infinity if for every number M, there is an integer N such that for all n larger than N, $a_n > M$.

If this is true, then we write $\lim_{n\to\infty} a_n = \infty$ or $a_n \to \infty$.

The sequence $\{a_n\}$ is said to diverge to negative infinity if for every number M, there is an integer N such that for all n larger than N, $a_n < M$.

If this is true, then we write $\lim_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$.

Group Exercise 8

For the "epsilon-delta" proof of convergence, we thought of that like a game (someone gives us an $\varepsilon > 0$, we have to show an N for which $\forall n > N$, $|a_n - L| < \varepsilon$). How would you describe the game for proving divergence to positive infinity?

Individual Exercise 9

Do you already know of a sequence that diverges to infinity? (Check the examples.)

Group Exercise 10

Can a sequence diverge without diverging to positive or negative infinity? If so, give an example.

In mathematics, a *counterexample* is a particular example that proves an assertion wrong. If you gave an example to the above question, it would be a counterexample to the (incorrect) assertion that "all divergent sequences diverge to positive or negative infinity". If someone had told me that all horses were brown, I could show a white horse as a counterexample to prove that person wrong.

Recap

We learned the following definitions:

- Sequence
- Term of a sequence
- Index of a term
- Set of natural numbers $\mathbb{N}$
- Fibonacci sequence
- Convergence of a sequence
- Limit of a sequence
- Divergence of a sequence
- Divergence to positive or negative infinity

We also learned the mathematical notions of:

- **Counterexample** to disprove an assertion
- **Quantifiers** to abbreviate words
- **Logical implications** that could be **reversible** or just **backward**

We will see these notions again and again! For this reason, I have posted links on Canvas to primers on each of those concepts, in case you need a refresher.

Homework

- Please find the syllabus module on Canvas, and complete the quiz there. This quiz will not contribute to your grade, but it is required before you complete Homework 1.
- Canvas Homework 1 has been posted, and will be due September 12 at 11:59 p.m.
- Written Homework 1 has also been posted, and will be due September 14. Get an early start on this, and make sure you ask me if you have any questions!