Lecture 10:
Lesson and Activity Packet

MATH 330: Calculus III

October 5, 2016

Announcements and Homework

- Written Homework due in class today
- Canvas Homework due Friday 11:59 p.m.
- No class Monday, October 10
- Exam 1 now Wednesday, October 12 (review on Friday)

Recap from last time

- Taylor series

Questions on any of this?

If not, then today's lesson will be more on **Taylor series**.

Recall:

Definition 1 (*Taylor series generated by $f(x)$ about $x = a$*)

The Taylor series generated by $f(x)$ about $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

We might ask (or we might be told to ask) how closely the Taylor polynomials approximate the function that generates it.

Theorem 1 (*Taylor's remainder formula*)

If *$f(x)$ has derivatives of all orders in some open neighborhood I of $x = a$,* ***then*** *for each $x \in I$,*

$$f(x) = \left[\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n\right] + R_N(x),$$

where $R_N(x)$ is called the ***remainder term*** *or the* ***error term****, and where for some c between a and x,*

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}.$$

Notes:

- The error $R_N(x)$ has the form of the other terms of the Taylor series, **except** that the derivative is evaluated at c, rather than at a
- c depends on x and on a

Once we know how closely the Taylor polynomials approximate f, we might also ask where (i.e., for which x) the Taylor series **converges** to f.

Theorem 2 (*Convergence of Taylor series*)

If $\lim_{n\to\infty} R_n(x) = 0$ *for all* $x \in I$*, then we say that the Taylor series* ***converges*** *to* f *on* I*, and we write*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example 1

Show that the Taylor series generated by $f(x) = e^x$ *at* $x = 0$ *converges to* $f(x)$ *for* ***every*** *real value of* x*.*

Taylor series convergence is important—a Taylor series is, after all, just a **power series**, and we have (had. . .) theorems:

- Termwise differentiation and integration on the interval of convergence
- Addition, subtraction, and multiplication of series on the intersection of their intervals of convergence

Example 2

Using The Taylor series for $\cos(x)$*, which we computed earlier and which we can prove converges to* $\cos(x)$ *for all* x*, write the first few terms of the Taylor series for* $\frac{1}{3}(2x + x\cos(x))$.

Group Exercise 1

Write the first few terms of the Taylor series for $e^x \cos(x)$.

Another use of convergent Taylor series is to express nonelementary integrals in series form.

Example 3

Express $\int \sin(x^2)\,\mathrm{d}x$ *as a power series.*

Yet another use: we can define exponentiation for imaginary numbers. Remember that previously, we had only defined it for real numbers! Recall:

Definition 2 (*Complex numbers*)

A complex number *is of the form* $a + bi$, *where* $a, b \in \mathbb{R}$ *and* $i := \sqrt{-1}$.

Note:

$$\begin{aligned} i^2 &= -1 \\ i^3 &= i^2(i) = -i \\ i^4 &= i^3(i) = (-i)(i) = 1 \\ i^5 &= i^4(i) = i \\ i^6 &= i^5(i) = i(i) = -1. \end{aligned}$$

Group Exercise 2

Substitute $x = i\theta$, $\theta \in \mathbb{R}$, *into the Taylor series for* e^x, *and separate into real and imaginary parts.*

Based on the results of the last exercise, we **define** $e^{i\theta} := \cos(\theta) + i\sin(\theta)$.

Group Exercise 3

Substitute $\theta = \pi$ *into the definition to compute* $e^{i\theta}$.

What you have just found is **Euler's identity**, a single equation that involves the five most important constants in mathematics.