Lecture 2:
Lesson and Activity Packet

MATH 330: Calculus III

September 9, 2016

Last time, we discussed:

- Sequence
- Term of a sequence
- Index of a term
- Set of natural numbers $\mathbb{N}$
- Fibonacci sequence
- Convergence of a sequence
- Limit of a sequence
- Divergence of a sequence
- Divergence to positive or negative infinity

We also learned the mathematical notions of:

- **Counterexample** to disprove an assertion
- **Quantifiers** to abbreviate words
- **Logical implications** that could be **reversible** or just **backward**

Questions on any of this?

The "epsilon" definition of sequence convergence is rigorous and applicable in all situations. But, as you noticed when studying the analogue definition for limits and continuity of functions in Calc I, it's not the easiest thing to work with. This is why mathematicians have used the rigorous definition to develop shortcuts; using these shortcuts to prove convergence of a sequence saves a great deal of effort.

Theorem 1 (*Theorem 55, p. 400*)

Let $\{a_n\}$ be a sequence of terms, and let $f(x)$ be a function whose domain is $\mathbb{R}^+$ ($\mathbb{R}^+$ denotes the set of all positive real numbers) such that $f(n) = a_n$ for all $n \in \mathbb{N}$. Then

$$\lim_{x\to\infty} f(x) = \lim_{n\to\infty} a_n,$$

provided that the limit of f exists.

Notes about this theorem:

- **"Provided that the limit of f exists"**.

  Group Exercise 1

  – *What is* $\lim_{x\to\infty} \cos(2\pi x)$*? [Does it exist?] Remember that x is a* continuous variable *in this limit; it can be any real number.*

  – *What is* $\lim_{n\to\infty} \cos(2\pi n)$*? Remember in this case that n is a* discrete variable*; that is, it can assume only integer values.*

- There are many sequences that don't have an associated function that defines their terms. (Can you think of an example?) Theorem 55 **does not apply** to those sequences. We would need other tools to compute their limits or to determine whether they converge at all.

- We actually knew very little about the concept "limit of a sequence" before this theorem, even if we remembered a lot of convenient rules about the "limit of a function" from our studies in Calc I. The reason Theorem 55 is so powerful is that by relating those two concepts, it suddenly gives us many more tools that can be applied to "limit of a sequence".

- It is not terribly difficult to prove this theorem: the main argument lies in unpacking what $f(x) \to L$ means in terms of "epsilsons" (just look up the definition from Calc I notes).

Group Exercise 2

You might remember the "sum rule" for computing the limit of a sum of functions:

> *If f and g are real-valued functions of a real variable, and if L and M are finite real numbers such that $f \to L$ and $g \to M$, then $f(x) + g(x)$ also converges, and $\lim\limits_{x\to\infty} [f(x) + g(x)] = L + M$.*

In other words,

> *The limit of the sum is the sum of the limits (provided that the limits exist).*

What are some other basic rules for computing function limits? (Draw from your knowledge of Calc I).

Group Exercise 3 (*3 minutes*)

Compute the limits of the following convergent sequences:

- $\left\{-\frac{1}{n}\right\}$
- $\left\{\frac{n-1}{n}\right\}$
- $\left\{\frac{4-7n^6}{n^6+3}\right\}$ *[Hint: Multiply this expression by* $\dfrac{1/n^6}{1/n^6}$, *which is equal to 1.]*

An important note about the sum rule: its **converse** does not hold. That is, if we know only that the sum of two sequences converges, we *cannot* conclude that each of the sequences also converges.

Example 1

$\{a_n\} := \{1, 2, 3, \ldots\}$ *diverges, and* $\{b_n\} := \{-1, -2, -3, \ldots\}$ *diverges. But their sum is* $\{a_n + b_n\} = \{0, 0, 0, \ldots\}$, *which* ***does*** *converge (to what?).*

We can even use the more sophisticated rules we know for computing limits of functions, like l'Hôpital's rule.

Group Exercise 4

Show that $\left\{\frac{\ln(n)}{n}\right\} \to 0$.

Theorem 2 (*Squeeze Theorem (Sandwich Theorem)*)

Let $\{a_n\}$, $\{b_n\}$, *and* $\{c_n\}$ *be sequences of real numbers. If* $a_n \leq b_n \leq c_n$ *holds for all* $n > N$ *(*N *is some index),* **and** *if* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n =: L$*, then* $\lim_{n\to\infty} b_n = L$ *too.*

Group Exercise 5

Use the Squeeze/Sandwich Theorem to compute the limits of the following sequences:

- $\left\{\dfrac{\cos(n)}{n}\right\}$
- $\left\{\dfrac{1}{2^n}\right\}$
- $\left\{\dfrac{(-1)^n}{n}\right\}$

Make sure that you state clearly what a_n, b_n, *and* c_n *are.*

Definition 1

A sequence $\{a_n\}$ *is* bounded from above *if there exists some* $M \in \mathbb{R}$ *such that for all* $n \in \mathbb{N}$, $a_n \leq M$. *In this case,* M *is called an* upper bound *for* $\{a_n\}$. *If* M *is an upper bound for* $\{a_n\}$ *but no number less than* M *is also an upper bound for* $\{a_n\}$*, then* M *is called the* least upper bound *for* $\{a_n\}$.

A sequence $\{a_n\}$ *is* bounded from below *if there exists some* $M \in \mathbb{R}$ *such that for all* $n \in \mathbb{N}$, $a_n \geq M$. *In this case,* M *is called a* lower bound *for* $\{a_n\}$. *If* M *is a lower bound for* $\{a_n\}$ *but no number greater than* M *is also a lower bound for* $\{a_n\}$*, then* M *is called the* greatest lower bound *for* $\{a_n\}$.

If $\{a_n\}$ *is bounded from above and from below, then* $\{a_n\}$ *is called* bounded. *If* $\{a_n\}$ *is not bounded, then we say that* $\{a_n\}$ *is* unbounded.

Group Exercise 6

Discuss whether the following sequences are bounded. Do they have upper or lower bounds?

- $\{1, 2, 3, \ldots, n, \ldots\}$
- $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1} \ldots\}$

Theorem 3

If a sequence converges, then it is bounded.

This is a *logical implication*, or an *if-then* statement. These statements are phrased "if A, then B" or "A implies B"[1]

Definition 2 (*Contrapositive*)

The contrapositive *of "if A then B" is "if not B, then not A".*

Group Exercise 7

What is the contrapositive of the statement, "If I am in Calc III, then I have already passed Calc II"?

Theorem 4

Every if-then statement is logically equivalent to its contrapositive (that is, the if-then statement is true exactly when the contrapositive is true, and is false exactly when the contrapositive is false).

Group Exercise 8

What is the contrapositive of "if a sequence converges, then it is bounded"?

[1] There are actually several ways that a logical implication can be written in English. "B, provided A" is another one. Try to think of more.

Definition 3 (*Converse*)

The converse of "if A then B" is "if B then A".

If-then statements are **not** logically equivalent to their converses. Sometimes they are true, and sometimes they are false.

Group Exercise 9

The statement "if I am in Calc III, then I have finished Calc II" has converse "if I have finished Calc II, then I am in Calc III." Is the converse true?

Group Exercise 10

What is the converse of "if a sequence converges, then it is bounded"? Is the converse true? Prove it, or provide a counterexample proving it false.

Group Exercise 11

What is the converse of "if $x + 3 = 7$, then $x = 4$"? Is the converse a true statement? Is the original implication true?

Definition 4 (*Monotonicity*)

- *A sequence* $\{a_n\}$ *is called* nondecreasing *if for all* $n \in \mathbb{N}$, $a_n \leq a_{n+1}$.
- *A sequence* $\{a_n\}$ *is called* nonincreasing *if for all* $n \in \mathbb{N}$, $a_n \geq a_{n+1}$.
- *A sequence* $\{a_n\}$ *is called* monotonic *if it is either nonincreasing, or nondecreasing.*

Group Exercise 12

Are the following sequences monotonic?

- $\{1, 2, \ldots, n, \ldots\}$
- $\{\frac{1}{2}, \frac{2}{3}, \ldots, \frac{n}{n+1}, \ldots\}$
- $\{1, \frac{1}{2}, \ldots, \frac{1}{2^n}, \ldots\}$
- $\{1, 1, \ldots, 1, \ldots\}$
- $\{1, -1, \ldots, (-1)^{n-1}, \ldots\}$

Theorem 5 (*Bounded Monotonic Sequences Converge*)

*If a sequence is **both** bounded and monotonic, then it converges.*

- If the sequence is nondecreasing, and bounded from above: must have a *least upper bound*, and that is the limit.
- If the sequence is nonincreasing, and bounded from below: must have a *greatest lower bound*, and that is the limit.

Group Exercise 13

What is the converse of the Bounded Monotonic Sequences Theorem? Is the converse true? Prove or provide a counterexample.

Recap

- Limits of sequences as limits of functions (when the functions exist)
- Sum/difference rule, product/quotient rule, constant multiple rule for sequences
- Using l'Hôpital's rule for computing sequence limits
- Squeeze/Sandwich theorem
- Boundedness
- Monotonicity
- Bounded Monotonic Sequences theorem

In addition, we learned a bit about logic:

- Logical implication / if-then statement
- Contrapositive
- Converse

Homework

- Please find the syllabus module on Canvas, and complete the quiz there. This quiz will not contribute to your grade, but it is required before you complete Lecture Modules 1 and 2.
- Canvas Homework 1 has been posted to Canvas, and will be due September 12.
- Written Homework 1 is available on Canvas and on paper in class today, and will be due September 14.