Lecture 5: Lesson and Activity Packet

MATH 330: Calculus III

September 19, 2016

Last time, we discussed:

- n^{th} term test for divergence
- Harmonic series
- Combining series
- Rehash of logical implication

Questions on any of this?

If not, then today's lesson will be more about infinite series, along with the **integral test** and the **comparison tests**.

Last time, we learned some formal notions that allow us to combine the terms of convergent series, and to factor out multiplication by a scalar—**for convergent series only**.

Two more basic principles that apply to both convergent **and** divergent series are:

Theorem 1 (*Adding/deleting finitely many terms*)

Adding or deleting finitely many terms from an infinite series does not change ***whether*** *the series converges (but will usually change the sum).*

Example 1

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \underbrace{\sum_{n=4}^{\infty} \frac{1}{5^n}}_{\text{“tail” of series}} .$$

So we can also write

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \frac{1}{5^n} - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}.$$

Group Exercise 1 (*30 seconds? 1 minute?*)

You know that $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges, and that its sum is 2. Add or subtract a finite number of terms to this series to obtain another infinite series that converges to 3. You can write this new series term-by-term, using a strategic “ $\cdots$ ” when appropriate.

Theorem 2 (*Reindexing*)

For $h > 0$, an infinite series can be reindexed as follows, without altering convergence:

$$\sum_{n=j}^{\infty} a_n = \underbrace{\sum_{n=j+h}^{\infty} a_{n-h}}_{\text{raises start index}} = \underbrace{\sum_{n=j-h}^{\infty} a_{n+h}}_{\text{lowers start index}} .$$

Example 2 (*Geometric series*)

- $\displaystyle\sum_{n=0}^{\infty} ar^n = a + ar^1 + ar^2 + ar^3 + \cdots$
- $\displaystyle\sum_{n=0+1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} ar^{n-1} = a + ar^1 + ar^2 + ar^3 + \cdots$
- $\displaystyle\sum_{n=0+7}^{\infty} ar^{n-7} = \sum_{n=7}^{\infty} ar^{n-7} = a + ar^1 + ar^2 + ar^3 + \cdots$

Group Exercise 2 (*2 minutes*)

Rewrite the geometric series $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}$ with starting index $n = 0$. Does this series converge? If so, what is its sum?

Recall that the partial sums S_n of an infinite series $\sum_{n=1}^{\infty} a_n$ are defined by the recursion formula

$$\begin{cases} S_1 := a_1, \\ S_n := S_{n-1} + a_n, \ n > 1. \end{cases}$$

Suppose that $\sum_{n=1}^{\infty} a_n$ has all terms $a_n \geq 0$. Then we have

$$S_1 \leq S_2 \leq S_3 \leq \cdots \leq S_n \leq S_{n+1} \leq \cdots .$$

In other words, the sequence $\{S_n\}$ is **nondecreasing**. We could also say it is **monotonically increasing**.

Group Exercise 3 (*1 minute*)

Rewrite the Monotonic Bounded Sequence Theorem here. If the terms a_n of $\sum_{n=1}^{\infty} a_n$ are all non-negative, then what does the Monotonic Bounded Sequence Theorem tell us about when $\{S_n\}$ converges? (Also recall: if $\{S_n\}$ converges as a sequence, then we say $\sum_{n=1}^{\infty} a_n$ converges as a series.)

So we have the following theorem:

Theorem 3

If $\sum_{n=1}^{\infty} a_n$ has $a_n \geq 0$ for all n (or for all except finitely many n), ***and*** *if $\{S_n\}$ is bounded from above, then $\sum_{n=1}^{\infty} a_n$ converges.*

Group Exercise 4 (*7 minutes*)

The converse of this statement is equivalent to:

> *If $\sum_{n=1}^{\infty} a_n$ has $a_n \geq 0$ for all but finitely many n,* ***and*** *if $\sum_{n=1}^{\infty} a_n$ converges, then $\{S_n\}$ is bounded from above.*

Is this converse true? No need to use the previous theorem here: just think about what it means, in terms of partial sums, for $\sum_{n=1}^{\infty} a_n$ to converge.

If a logical implication and its converse are **both** true, then we call the statement an "equivalency", or an "if-and-only-if" statement, which is sometimes abbreviated "iff" or written with the implication sign $\iff$.

Theorem 4

A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms converges ***if and only if*** *its partial sums are bounded from above.*

This is what we just showed.

Let's revisit the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{>\frac{8}{16}=\frac{1}{2}} + \cdots.$$

The first two terms sum to $\frac{3}{2}$.
The next two terms' sum is greater than $\frac{1}{2}$.
The next four terms' sum is greater than $\frac{1}{2}$.
The next eight terms' sum is greater than $\frac{1}{2}$.
$\vdots$
The next 2^n many terms' sum is greater than $\frac{1}{2}$.

Group Exercise 5 (*5 minutes*)

*Find a **lower** bound on the $(2^n)^{th}$ partial sum of the harmonic series. Can the partial sums have an upper bound?*

Group Exercise 6 (*5 minutes*)

Does the harmonic series converge? Use the theorem from the previous page.

You have just proven that **the harmonic series does not converge**! (So gratifying.) Another interesting question might be:

Example 3

Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converge?*

Theorem 5 (*Integral Test*)

*Suppose $\{a_n\}$ is a sequence of positive terms, and $a_n = f(n)$ with f a continuous, positive, decreasing function for all $x > N \in \mathbb{N}$. Then $\displaystyle\sum_{n=N}^{\infty} a_n$ and $\displaystyle\int_N^{\infty} f(x)\ \mathrm{d}x$ either **both converge**, or **both diverge**[a].*

[a] This notion is sometimes written, "the sum and the integral **converge or diverge together**".

So to prove that a series converges, show its terms are continuous, positive, and decreasing (perhaps after a finite number of terms), and then prove that the improper integral converges. To prove that a series diverges, show its terms are continuous, positive, and decreasing (perhaps after a finite number of terms), and then prove that the improper integral diverges. **Important**: This theorem doesn't tell us the **sum** of a series: just **whether** it converges! The sum of the series **is not equal to** the improper integral.

In addition to the *geometric series*, there is another important specific type of series to know about: the **p-series**.

Definition 1 (*p-Series*)

A p-series is of the form $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{an+b}\right)^p$, where $p > 0$ is constant, and a and b are real numbers[a].

[a] Sometimes written: $a, b \in \mathbb{R}$.

Example 4

Consider for a moment the case where $a = 1$ and $b = 0$. That is, consider $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Does this converge, or diverge? Does it depend on p?

So we've learned the following addendum to the definition:

Definition 2 (*p-Series*)

A p-series is of the form $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{an+b}\right)^p$*, where* $p>0$ *is constant, and* $a,b\in\mathbb{R}$*. A p-series converges only when* $p>1$*. If* $p\leq 1$*, the series diverges.* ***Important****: This theorem doesn't tell us the* ***sum*** *of a convergent p-series—just that it converges.*

Individual Exercise 7

What are the values of a, b, and p in the following p-series?

- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(3+5n)^{10}}$

Sometimes we know a series converges, but we don't know what it converges **to**. We can estimate it using the sequence of partial sums, of course, which converge to the sum of the infinite series. But there is always a remainder error in such an estimate. To wit:

$$\left(\sum_{n=1}^{\infty}a_n\right)-S_N=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{N}a_n=\sum_{n=N}^{\infty}a_n=a_{N+1}+a_{N+2}+\cdots.$$

Definition 3 (*Remainder Term*)

The remainder left over when the N^{th} *partial sum is used to approximate the sum of a convergent infinite series is* $R_N:=a_{N+1}+a_{N+2}+\cdots$.

We can use integrals to put upper and lower bounds on this remainder term!

$$\int_{N+1}^{\infty}f(x)\ \mathrm{d}x\leq R_N\leq\int_{N}^{\infty}f(x)\ \mathrm{d}x,$$

where $f(n)=a_n$ for all $n\in\mathbb{N}$.

Example 5

Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *using* $n = 10$.

Recap

- Adding/subtracting finitely many terms
- Reindexing
- Integral test
- Proving harmonic series diverges
- p-series
- Error estimates/remainders of partial sums

Homework

- Canvas Homework 3 due 11:59 p.m. Tuesday.
- Written Homework 2 due at the beginning of class on Friday (this is a change from the schedule!).