Lecture 8:
Lesson and Activity Packet

MATH 330: Calculus III

September 30, 2016

Announcements and Homework

- Written Homework due in class on Wednesday next week
- Canvas Homework due Monday 11:59 p.m.
- Exam 1 next Friday, October 7 (review day on Wednesday)

Recap

- Review of series until now
- Root test
- Ratio test
- Alternating series
- Absolute vs. conditional convergence

Questions on any of this?

If not, then today's lesson will be on **power series**.

Definition 1 (*Power series about $x = a$*)

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots ,$$

in which the center a is a constant, and the coefficients c_0, c_1, ... may depend on n.

Notes:

- Often have $a = 0$: $\sum^{i} nfty_{n=0} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$
- Think of these like “infinite polynomials” —we will figure out how to manipulate them (e.g., adding, subtracting, differentiating, integrating) like “finite” polynomials

Example 1

If $a = 0$ and all coefficients c_n are 1, then the power series is

$$\sum_{n=0}^{\infty} x^n.$$

Group Exercise 1 (*5 minutes*)

The power series $\sum_{n=0}^{\infty} x^n$ is also classifiable as what kind of series? For which values of x does the series converge? What does it converge to in that case?

So we found:

For $x \in (-1,1)$, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

We can equivalently write:

For $x \in (-1,1)$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

This latter representation might make it more clear that for $x \in (-1,1)$, a **polynomial approximation** of $\frac{1}{1-x}$ can be obtained by taking **partial sums** of the power series $\sum_{n=0}^{\infty} x^n$:

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= 1 + x \\
P_2(x) &= 1 + x + x^2 \\
P_3(x) &= 1 + x + x^2 + x^3 \\
&\vdots \\
P_N(x) &= \sum_{n=0}^{N} x^n \\
&\vdots
\end{aligned}$$

These polynomial approximations get closer to $\frac{1}{1-x}$ as $N \to \infty$.

Example 2

We know that $\frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = 2$. *Watch the partial sums of the series* $\sum_{n=0}^{\infty}(\frac{1}{2})^n$ *get closer and closer to 2:*

$$\left\{P_0\left(\frac{1}{2}\right), P_1\left(\frac{1}{2}\right), P_2\left(\frac{1}{2}\right), P_3\left(\frac{1}{2}\right), \ldots, P_N\left(\frac{1}{2}\right), \ldots\right\} = \left\{1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots, \frac{2^{N+1}-1}{2^N}, \ldots\right\}$$

Group Exercise 2 (*5 minutes*)

In the definition of the power series, take $a = 2$ and $c_n = \left(-\frac{1}{2}\right)^n$:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = \sum_{n=0}^{\infty} \left(-\frac{x-2}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{2-x}{2}\right)^n .$$

For which values of x does this series converge? What does the series converge to, in that case?

Group Exercise 3 (*3 minutes*)

Give the general form of a polynomial approximation of the function $\frac{2}{x}$, and state where this approximation is valid.

We are often interested in the question: **For which values of x does a given power series converge?**

In each of the two previous examples/exercises, we saw that the power series centered at a converged for x **within a certain radius of** a. We'll see that this was no coincidence.

Theorem 1 (*Power Series Convergence*)

The convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ *is described by one of the following cases:*

Case I. *There exists $R > 0$ such that the series converges absolutely for $|x-a| < R$ (i.e., converges for $x \in (a-R, a+R)$), but diverges if $|x-a| > R$ (i.e., diverges for $x \notin (a-R, a+R)$). At the endpoints $\{a-R, a+R\}$, the series* ***may or may not*** *converge absolutely; these must be tested individually.*

Case II. *The series converges absolutely for all x. In this case, we say $R = \infty$.*

Case III. *The series converges* ***only*** *at $x = a$, and diverges elsewhere. In this case, we say $R = 0$.*

Notes:

- R is called the **radius of convergence** and the interval where the series converges is called the **interval of convergence**. The interval of convergence may look like any of the following:

  1. $(a-R, a+R)$
  2. $(a-R, a+R]$
  3. $[a-R, a+R)$
  4. $[a-R, a+R]$
  5. $(-\infty, +\infty)$
  6. $\{a\}$

Group Exercise 4 (*8 minutes*)

State the radius and interval of convergence for the following power series:

- $\displaystyle\sum_{n=0}^{\infty} x^n$
- $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2-x}{2}\right)^n$
- $\displaystyle\sum_{n=0}^{\infty} n^n x^n$ *[Hint: Use the ratio test.]*
- $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Convergence is important for multiplying power series, and for termwise differentiation and integration.

Theorem 2 (*Series multiplication for power series*)

If $A(x) := \sum a_n x^n$ *and* $B(x) := b_n x^n$ *both converge absolutely for* $|x| < R$,

Then *for* $c_n := \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0$, *the series* $\sum_{n=0}^{\infty} c_n x^n$ *converges absolutely to* $A(x)B(x)$ *for* $|x| < R$. *That is,*

$$\left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum c_n x^n, \quad |x| < R.$$

Notes:

1. Take $R := \min\{R_a, R_b\}$, where R_a and R_b are the radii of convergence of $A(x)$ and $B(x)$ respectively.
2. Computing the c_n is often tedious!(!!) Can be easier to restrict the computation to the first few terms and use strategic $(\cdots)$'s.

Theorem 3 (*Termwise differentiation of power series*)

If $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then it defines a function $f(x) := \sum c_n(x-a)^n$ for x such that $|x-a| < R$, and:

$$
\begin{aligned}
f'(x) &= \sum nc_n(x-a)^{n-1} \\
f''(x) &= \sum n(n-1)c_n(x-a)^{n-2} \\
f'''(x) &= \sum n(n-1)(n-2)c_n(x-a)^{n-3} \\
&\vdots
\end{aligned}
$$

Each of the series for the derivatives converges on the same interval, where $|x-a| < R$.

The general idea: if a power series converges, then it can be differentiated termwise on the interval of convergence.

Example 3

$f(x) := \frac{1}{1-x}$

This may **not** work for series that are not power series!

Example 4

$$\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$$

Theorem 4 (*Termwise integration of power series*)

If $\sum c_n(x-a)^n$ *has a radius of convergence* $R > 0$*, then it defines a function* $f(x) := \sum c_n(x-a)^n$ *for* x *such that* $|x-a| < R$*, and* $\int f(x)\,\mathrm{d}x = \sum c_n \frac{(x-a)^{n+1}}{n+1} + C$ *on the interval of convergence.*

General idea: if a power series converges, then it can be integrated termwise on the interval of convergence.