Lecture 9:
Lesson and Activity Packet

MATH 330: Calculus III

October 3, 2016

Announcements and Homework

- Written Homework due in class on Wednesday
- Canvas Homework due tonight 11:59 p.m.
- Exam 1 now Wednesday, October 12 (review on Friday this week)

Recap from last time

- Power series
- Radius and interval of convergence
- Multiplying power series
- Differentiating power series
- Integrating power series

Questions on any of this?

If not, then today's lesson will be on **Taylor series**, one particularly important kind of power series.

Recall that if a function $f(x)$ can be expressed as a power series centered at $x = a$ that has radius R of convergence, then $f(x)$ is infinitely differentiable, and the finite sum

$$P_N(x) := \sum_{n=0}^{N} c_n (x-a)^n$$

is a polynomial that approximates $f(x)$ more and more closely as $N \to \infty$.

Now, for something much stronger:

Theorem 1

Every function *$f(x)$ that is* ***infinitely differentiable*** *at $x = a$ can be written as a power series (with a nonzero radius of convergence).*

Definition 1 (*Taylor series generated by f at $x = a$*)

Let $f(x)$ be infinitely differentiable in some neighborhood of $x = a$ (that is, in some open interval $(a - R, a + R)$). Then the Taylor series generated by f at $x = a$ *is:*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots . \end{aligned}$$

If $a = 0$, we refer to the series as the Maclaurin series generated by f.

Example 1

Find the Taylor series generated by $f(x) = \frac{1}{x}$ *at* $a = 2$. *Where (if anywhere) does it converge to* $1/x$?

Definition 2 (*Taylor polynomials*)

A Taylor polynomial of order N *is (usually) the* N^{th} *partial sum of the Taylor series:*

$$P_N(x) := \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N.$$

Example 2

Find the Taylor series and the fourth-order Taylor polynomial generated by $f(x) = e^x$ *at* $x = 0$*. [This is the same question as "Find the Maclaurin series generated by* $f(x) = e^x$*, and the fourth-order Taylor polynomial approximation of* f *about* $x = 0$*.]*

Group Exercise 1 (*10 minutes*)

Show that the Maclaurin series generated by $\cos(x)$ *is* $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. *Where does this series converge to* f*?*

On the previous page, we showed that $f(x) = \cos(x)$ has the following derivatives, with their evaluations at $x = 0$:

$$\begin{aligned} f(x) &= \cos(x) & f(0) &= 1 \\ f'(x) &= -\sin(x) & f'(0) &= 0 \\ f''(x) &= -\cos(x) & f''(0) &= -1 \\ f'''(x) &= \sin(x) & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos(x) & f^{(4)}(0) &= 1 \\ &\vdots \end{aligned}$$

The polynomial approximations are therefore:

$$\begin{aligned} &P_0(x) = 1 \\ &P_1(x) = 1 + 0 = 1 \\ &P_2(x) = 1 - \frac{x^2}{2} \\ &P_3(x) = 1 - \frac{x^2}{2} + 0 = 1 - \frac{x^2}{2} \\ &P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} \\ &P_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + 0 = 1 - \frac{x^2}{2} + \frac{x^4}{4!} \\ &\quad\vdots \end{aligned}$$

Notice that the N^{th} order (not "degree"!) Taylor polynomial approximation of $\cos(x)$ is **not** the same as the N^{th} partial sum of the Maclaurin series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, because in this representation, the series "skips" the odd-numbered terms whose value is zero. Be careful!

Group Exercise 2

Find the Taylor polynomials of order 1, 2, and 3 for $f(x) := \sin(x)$ *about* $x = 0$.

Group Exercise 3

Find the Taylor series generated by $f(x) := \frac{1}{x^2}$ *about* $x = 1$.