Lec. 21 : Oct. 21, 2016.

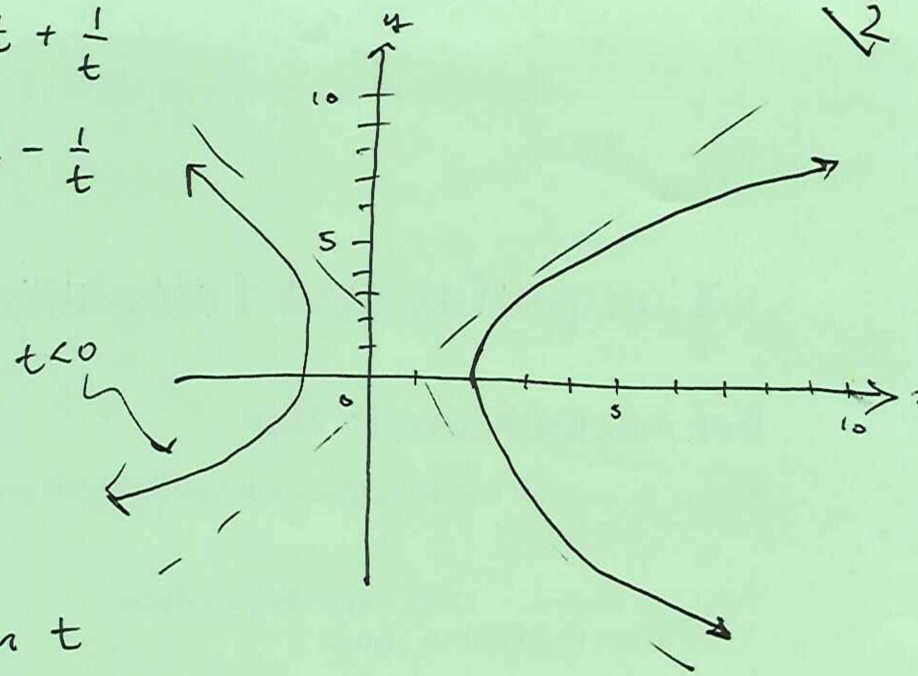
Housekeeping:
- Canvas Homework 11:59 p.m. tonight
- Written homework in class Weds.

Last time: Existence & Uniqueness of Parametrization

Questions?

This time: Calculus of Parametric curves.

<u>Last warmup</u> :
$$\begin{cases} x = t + \frac{1}{t} \\ y = t - \frac{1}{t} \\ t > 0 \end{cases}$$



- Eliminate the parameter t
- Identify the shape of the graph

<u>Hint</u> : Compute $x+y$; compute $x-y$

$$\begin{array}{r} x+y = t + \frac{1}{t} \\ +\left(t - \frac{1}{t}\right) \\ \hline 2t \end{array} \qquad \begin{array}{r} x-y = t + \frac{1}{t} \\ -\left(t - \frac{1}{t}\right) \\ \hline \frac{2}{t} \end{array}$$

So, $\quad x+y = 2t \quad$ and $\quad x-y = \frac{2}{t}$.

$\quad t = \frac{x+y}{2} \quad$ and $\quad t = \frac{2}{x-y}$.

Then $\quad \frac{x+y}{2} = \frac{2}{x-y} \Rightarrow (x+y)(x-y) = 2 \cdot 2$

$$x^2 - y^2 = 4.$$

This eq'n is of a hyperbola.

Calculus of Param. Curves.

- Derivatives

Recall: Chain rule. $$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Then $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \text{provided } \frac{dx}{dt} \neq 0.$$

- Second derivatives.

If $$\frac{d}{dx}[y] = \frac{\frac{d}{dt}[y]}{dx/dt},$$

Then $$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

Example ①. Find the eq'n of the line tangent to the curve

$$\begin{cases} x = \sec(t) \\ y = \tan(t) \\ -\frac{\pi}{2} < t < \frac{\pi}{2} \end{cases}$$

, at the point $(\sqrt{2}, 1)$, where $t = \pi/4$.

Find $\left.\frac{dy}{dx}\right|_{(\sqrt{2},1)}$ at the pt. or $t=\pi/4$. Well, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

$$\frac{dy}{dt} = \frac{d}{dt}\left[\tan(t)\right] = \frac{d}{dt}\left[\frac{\sin(t)}{\cos(t)}\right] = \frac{\cos(t)\frac{d}{dt}[\sin t] - \sin t \cdot \frac{d}{dt}[\cos t]}{\cos^2 t}$$

$$= \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t.$$

$$\frac{dx}{dt} = \frac{d}{dt}\left[\sec t\right] = \frac{d}{dt}\left[\frac{1}{\cos t}\right] = \frac{d}{dt}\left[(\cos t)^{-1}\right] =$$

$$= (-1)(\cos t)^{-2}\left(\frac{d}{dt}(\cos t)\right) = \frac{\sin t}{\cos^2 t} =$$

$$= \sec(t) \cdot \tan(t).$$

So $$\frac{dy}{dx} = \frac{\sec^2 t}{\sec t \cdot \tan t} = \frac{\sec t}{\tan t} = \frac{1}{\cos t} \cdot \frac{\cos t}{\sin t} = \frac{1}{\sin t} =$$

$$= \csc t.$$

Ex ① ct'd.

If $\frac{dy}{dx} = \csc(t)$, then $\left.\frac{dy}{dx}\right|_{t=\pi/4} = \csc(\pi/4) = \frac{1}{\sin(\pi/4)}$

$= \frac{1}{\sqrt{2}/2}$

$= \frac{2}{\sqrt{2}} = \sqrt{2}.$

So our line tangent to $\begin{cases} x = \sec t \\ y = \tan t \\ -\frac{\pi}{2} < t < \frac{\pi}{2} \end{cases}$ at $(\sqrt{2}, 1)$ has slope $\sqrt{2}$.

$$y - 1 = \sqrt{2}\,(x - \sqrt{2}) = \sqrt{2}\,x - 2$$

$$\Rightarrow \boxed{y = \sqrt{2}\,x - 1.}$$

Example ② Find $\dfrac{d^2y}{dx^2}$ as a fn. of t if:

$$\begin{cases} x = t - t^2 \\ y = t - t^3 . \end{cases}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}, \quad \text{and} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left[t - t^2\right] = 1 - 2t .$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[t - t^3\right] = 1 - 3t^2 .$$

So $\dfrac{dy}{dx} = \dfrac{1-3t^2}{1-2t}$, and

$$\frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{(1-2t)\frac{d}{dt}\left[1-3t^2\right] - (1-3t^2)\cdot\frac{d}{dt}\left[1-2t\right]}{(1-2t)^2}$$

$$= \frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^2}$$

$$= \frac{-6t + 12t^2 + 2 - 6t^2}{(1-2t)^2} = \frac{6t^2 - 6t + 2}{(1-2t)^2} .$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} = \frac{\left(\frac{6t^2-6t+2}{(1-2t)^2}\right)}{1-2t} = \boxed{\frac{6t^2-6t+2}{(1-2t)^3}} .$$

Procedure for 2^{nd} deriv. was:

1. Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

2. Find $\frac{dy}{dx} = \frac{dx/dt}{dy/dt}$

3. Differentiate (2) to find $\frac{d}{dt}\left[\frac{dy}{dx}\right]$.

4. Use $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[{dy}/{dx}\right]}{{dx}/{dt}}.$$