Lecture 3: Lesson and Activity Packet

MATH 330: Calculus III

September 12, 2016

Last time, we discussed:

- Limits of sequences as limits of functions (when the functions exist)
- Sum/difference rule, product/quotient rule, constant multiple rule for sequences
- Using l'Hôpital's rule for computing sequence limits
- Squeeze/Sandwich theorem
- Boundedness
- Monotonicity
- Bounded Monotonic Sequences theorem

In addition, we learned a bit about logic:

- Logical implication / if-then statement
- Contrapositive
- Converse

Questions on any of this?

If not, then on today's agenda is *infinite series.*

Addition is what we call in mathematics a *binary operator*. It takes exactly two input values, and *operates* on them to produce an output value.

Group Exercise 1 (*30 seconds*)

What are some other binary operators?

Subtrac'n, mult., division; set operations (union, intersec'n, etc.)

Group Exercise 2 (*1 minute*)

It's true: addition always takes **exactly** *two input values. How, then, do you make sense of an expression like this one?*

$$3+\Big(2+\big(17+(1+7)\big)\Big)$$

We found a way to sum 5 numbers above without violating the definition of addition, but what about summing **infinitely many** numbers? We would not never be able to stop drawing the "opening" parenthesis (, because we would never get to the "last" two numbers in the list!

Nevertheless, such questions might arise naturally.

Example 1

Our first approximations of the area under the graph *of a function were the (finite) sums of the areas of rectangles:*

The more rectangles we had, the more "accurate" our Riemann sum was. ***The Riemann integral was the limit of those sums.***

So we would like a way to define an "infinite sum" that is consistent with what we know to be true about "finite sums".

Let's begin with an example. Consider the sequence $\{a_n\}$, where $a_n := \frac{1}{2^n}$.

Individual Exercise 3 (*30 seconds*)

Write the first 5 terms of $\{a_n\}$.

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

Define another sequence $\{S_n\}$, where

$$S_n := \sum_{k=1}^{n} a_k = \sum_{n=1}^{n} \frac{1}{2^k}.$$

Group Exercise 4 (*2 minutes*)

The first two terms of $\{S_n\}$ are

$$S_1 = a_1 = \frac{1}{2}$$

and

$$S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Compute the next three terms S_3, S_4, and S_5.

$S_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$

$S_4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$

$S_5 = \frac{15}{16} + \frac{1}{32} = \frac{31}{32}$

Group Exercise 5

Write your answers to the above exercise in the following blanks to form a true statement:

$$\left\{\sum_{k=1}^{n} \frac{1}{2^k}\right\} = \left\{S_1, S_2, S_3, S_4, S_5, \ldots\right\}$$

$$= \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots\right\}$$

Group Exercise 6 (*2 minutes*)

Confirm that

$$S_1 = \frac{2^1-1}{2^1}, \quad S_2 = \frac{2^2-1}{2^2}, \quad S_3 = \frac{2^3-1}{2^3}, \quad S_4 = \frac{2^4-1}{2^4}, \quad \textit{and} \quad S_5 = \frac{2^5-1}{2^5}.$$

$S_1 \overset{?}{=} \frac{2^1-1}{2^1} = \frac{2-1}{2} = \frac{1}{2}$ ✓

$S_2 \overset{?}{=} \frac{2^2-1}{2^2} = \frac{4-1}{4} = \frac{3}{4}$ ✓

$S_3 \overset{?}{=} \frac{2^3-1}{2^3} = \frac{8-1}{8} = \frac{7}{8}$ ✓

$S_4 \overset{?}{=} \frac{2^4-1}{2^4} = \frac{16-1}{16} = \frac{15}{16}$ ✓

$S_5 \overset{?}{=} \frac{2^5-1}{2^5} = \frac{32-1}{32} = \frac{31}{32}$ ✓

Group Exercise 7 (*1 minutes*)

Based on the above exercise, what would you suspect is a general formula for S_n?

$S_n = \frac{2^n - 1}{2^n}$

On Written Homework 2 you will prove rigorously that when $a_n = \frac{1}{2^n}$, a general formula for S_n is

$$S_n = \frac{2^n - 1}{2^n}.$$

For now, you are allowed to just take this for granted.

So in the beginning, we defined the terms S_n of our new sequence as $\sum_{k=1}^{n} \frac{1}{2^k}$, and just now, we figured out a simpler expression: $S_n = \frac{2^n-1}{2^n}$. The latter expression is much easier to work with using the standard tools we know for sequence convergence.

Group Exercise 8 (*2 minutes*)

Does the sequence $\{\frac{2^n-1}{2^n}\}$ converge? If so, what is its limit?

$$\lim_{n\to\infty} \frac{2^n-1}{2^n} = \lim_{n\to\infty} \frac{2^n-1}{2^n}\left(\frac{2^{-n}}{2^{-n}}\right)$$

$$= \lim_{n\to\infty} \frac{1 - 1/2^n}{1}$$

$$= \lim_{n\to\infty} 1 - \frac{1}{2^n} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1 - 0$$

$$= 1.$$

Group Exercise 9 (*1 minute*)

Does the sequence $\{\sum_{k=1}^{n} \frac{1}{2^k}\}$ converge? If so, what is its limit? Please write your answer in the blank below. [Hint: Use your answer to the last exercise.]

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \underline{\ 1\ }$$

On the last page, you should have found that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = 1.$$

Remember that our original task was to find a way to define the sum of **infinitely many** numbers. We don't yet have a formal definition, but the previous examples just helped us formulate a condition: **when we apply our future definition of an "infinite sum" to the particular example $\sum_{k=1}^{\infty} \frac{1}{2^k}$, we want it to show that**

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1,$$

or in other words,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

Definition 1 (***Infinite series, n^{th} partial sum, convergence, divergence***)

Given a sequence $\{a_k\}$ of numbers, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_k + \cdots$$

is called an infinite series. *The number a_k is the k^{th}* term *of the series. Sometimes an infinite series is written*

$$\sum_{k=1}^{\infty} a_k.$$

The sequence $\{S_n\}$ whose terms are defined by the finite sums:

$$S_1 := a_1$$
$$S_2 := a_1 + a_2$$
$$S_2 := a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n := a_1 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is called the sequence of partial sums *of the infinite series, the number S_n being called the n^{th}* partial sum. *If the sequence of partial sums converges to a limit L, we say that the* **series** converges *and that its sum is L. We write,*

$$\sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums does not converge, then we say that the **series** diverges.

Group Exercise 10

The series $\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges. Show this by finding the sequence of partial sums.

$S_1 = (-1)^{1+1} = (-1)^2 = 1$

$S_2 = S_1 + (-1)^{1+2} = 1 - 1 = 0$

$S_3 = S_2 + (-1)^{1+3} = 0 + 1 = 1$

$S_4 = S_3 + (-1)^{1+4} = 1 - 1 = 0$

So $S_n = 1 + (-1)^{n+1}$, or $\{S_n\} = \{1, 0, 1, 0, \ldots\}$.

Doesn't converge.

Group Exercise 11

Show that the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges, and find its sum.

$S_1 = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$

$S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$

$S_3 = 1 \underbrace{- \frac{1}{2} + \frac{1}{2}}_{0} \underbrace{- \frac{1}{3} + \frac{1}{3}}_{0} - \frac{1}{4} = 1 - \frac{1}{4}$

$S_4 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} = 1 - \frac{1}{5}$

$S_n = 1 - \frac{1}{n}$. So $\{S_n\} \to 1$ and $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$.

The sum in this example was called a **telescoping sum**. The sums in the very first motivating example, and in Exercise 10, are called **geometric series**.

Definition 2 (*Geometric Series*)

A geometric series *is of the form*

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots,$$

which is equivalent to

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^n + \cdots.$$

For a given geometric series:

- *If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.*
- *If $|r| \geq 1$ and $a \neq 0$, then $\sum_{n=0}^{\infty} ar^n$ diverges.*

Group Exercise 12 (*2 minutes*)

Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

Is this a geometric series? If so, what is a and what is r, and does it converge? If so, to what? Does this contradict anything we showed in the example where we computed the partial sums of this series?

Yes. $a = 1$, $r = \frac{1}{2}$.

Converges, because $|r| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$.

Converges to $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$.

Doesn't contradict $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, bc. of $n = 0$ term.

Group Exercise 13 (*3 minutes*)

Consider the series

$$10 - \frac{10}{3} + \frac{10}{9} - \frac{10}{27} + \cdots = \sum_{n=0}^{\infty} 10\left(-\frac{1}{3}\right)^n .$$

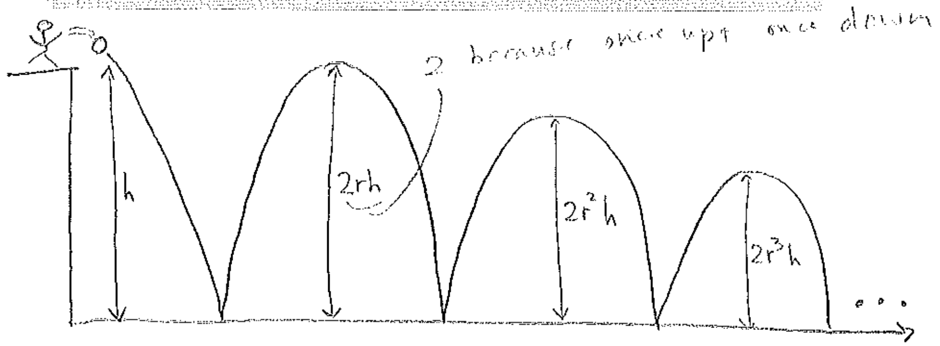
Is this a geometric series? If so, what is a and what is r, and does it converge? If so, to what?

$a = 10$, $r = -\frac{1}{3}$. $|r| = |-\frac{1}{3}| = \frac{1}{3} < 1$, so converges.

Conv. to $\frac{a}{1-r} = \frac{10}{1-(-\frac{1}{3})} = \frac{10}{4/3} = \frac{30}{4} = 7.5$

Group Exercise 14

You drop a ball from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds by a positive distance rh, where r is positive but less than 1. Find the total distance that the ball travels up and down.



Total dist. $= h + 2rh + 2r^2h + 2r^3h + \cdots$

$$= h + \sum_{m=1}^{\infty} 2hr^m = h + \left(\sum_{m=0}^{\infty} 2hr^m\right) - 2h$$

$$= -h + \frac{2h}{1-r}$$

$$= \frac{-h(1-r)+2h}{1-r}$$

$$= \frac{h+r}{1-r}.$$

Example 2

Express the repeating decimal $5.23\overline{23}$ as the ratio of two integers.

To begin, observe that

$$\begin{aligned} 5.23\overline{23} &= 5 + \frac{23}{100} + \frac{23}{10,000} + \frac{23}{1,000,000} + \cdots \\ &= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots \\ &= 5 + \frac{23}{100}\left(1 + \frac{1}{(100)^2} + \frac{1}{(100)^3} + \cdots\right) \\ &= 5 + \frac{23}{100}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n . \end{aligned}$$

For this geometric series, $a = 1$, $r = 1/100$, and since $|r| < 1$, we know the series converges; in particular, it converges to

$$5 + \frac{23}{100}\left(\frac{1}{1-\frac{1}{100}}\right) = 5 + \frac{23}{100}\cdot\frac{100}{99} = 5 + \frac{23}{99} = \frac{518}{99}.$$

Recap

- Series definition
  - Partial sums
  - Series convergence
  - Sum of series
  - Series divergence
- Telescoping series
- Geometric series
  - Repeating decimals

Homework

- Canvas Homework 1 due 11:59 p.m. tonight.
- Written Homework 1 due at the beginning of class on Wednesday.
- Module 3 is on Canvas; not required, but the modules contain useful supplemental information and links.