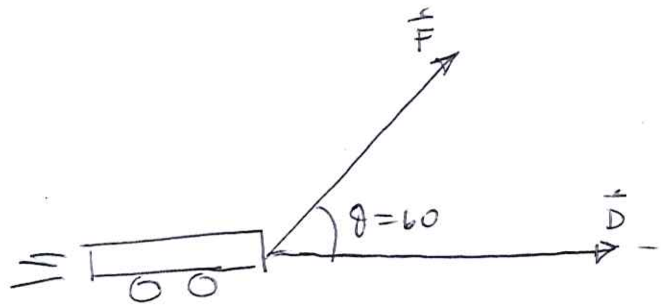
DEC. 7, 2016

The <u>work</u> done by a constant force $\vec{F}$ acting through a displacement $\vec{D}$ is

$$W := \vec{F} \cdot \vec{D}$$

<u>Ex.</u> If $|\vec{F}| = 40\text{ N}$, $|\vec{D}| = 3\text{ m}$, and $\theta = 60°$



$$W = \vec{F} \cdot \vec{D}$$

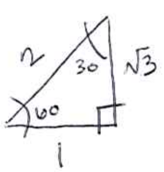
$$W = |\vec{F}||\vec{D}| \cos\theta$$

$$= (40\text{N})(3\text{m}) \cos(60°)$$

$$= (40\text{ N})(3\text{ m})\left(\tfrac{1}{2}\right)$$

$$= 60\text{ N}\cdot\text{m}$$

$$= 60\text{ J}$$



If $\theta = \arccos\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$ for any vectors $\vec{u}, \vec{v}$

Then $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

$$\boxed{\cos\theta\, |\vec{u}||\vec{v}| = \vec{u} \cdot \vec{v}}$$

DEF. The cross product $\vec{u} \times \vec{v}$ is the vector

$$\vec{u} \times \vec{v} := \left(|\vec{u}||\vec{v}| \sin\theta \right) \hat{n},$$

where $\hat{n}$ is the (unit) vector normal (= orthogonal) to both $\vec{u}$ and $\vec{v}$, chosen by the right-hand rule, and θ is the angle btwn. $\vec{u}$ and $\vec{v}$.

THIS IS A VECTOR.

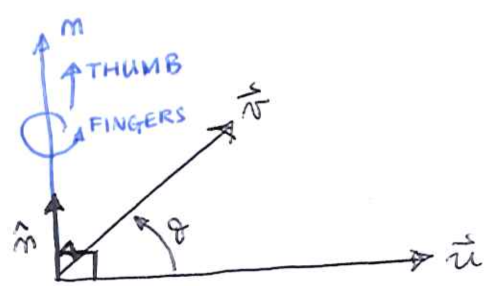
THE CROSS PRODUCT OF TWO VECTORS IS A

VECTOR.

THIS IS WHY THE CROSS PRODUCT IS SOMETIMES CALLED...

... THE VECTOR PRODUCT

The normal vector $\hat{n}$:



Note: $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$

The "standard" properties (commutativity, distributivity, etc.) DO NOT NECESSARILY HOLD FOR THE CROSS PROD.

The cross product can be computed as the determinant of a symbolic matrix:

Recall: $$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - cb$$

$$\det\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right) = a_1 \det\left(\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}\right) - a_2 \det\left(\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}\right) + a_3 \det\left(\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}\right)$$

$$= -b_1 \det\left(\begin{bmatrix} a_2 & a_3 \\ c_2 & c_3 \end{bmatrix}\right) + b_2 \det\left(\begin{bmatrix} a_1 & a_3 \\ c_1 & c_3 \end{bmatrix}\right) - b_3 \det\left(\begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix}\right)$$

$$= c_1 \det\left(\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}\right) - c_2 \det\left(\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}\right) + c_3 \det\left(\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right)$$

"cofactor expansion"

The cross prod. as a determinant:

$$\vec{u} \times \vec{v} = \det\left(\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\right)$$

$$= \hat{\imath} \det\left(\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}\right) - \hat{\jmath} \det\left(\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}\right) + \hat{k} \det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right)$$

$$= \hat{\imath}\,(u_2 v_3 - u_3 v_2) - \hat{\jmath}\,(u_1 v_3 - u_3 v_1) + \hat{k}\,(u_1 v_2 - u_2 v_1)$$

Example. Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ if $\vec{u} = \langle 2, 1, 1 \rangle$
$\vec{v} = \langle -4, 3, 1 \rangle$.

$$\vec{u} \times \vec{v} = \det\left(\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{bmatrix}\right)$$

$$= \hat{i} \det\left(\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}\right) - \hat{j} \det\left(\begin{bmatrix} 2 & 1 \\ -4 & 1 \end{bmatrix}\right) + \hat{k} \det\left(\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}\right)$$

$$= \hat{i}\,(1\cdot1 - 3\cdot1) - \hat{j}\,(2\cdot1 - (-4)\cdot1) + \hat{k}\,(2\cdot3 - (-4)\cdot1)$$

$$= \hat{i}\,(1-3) - \hat{j}\,(2+4) + \hat{k}\,(6+4)$$

$$= -2\hat{i} - 6\hat{j} + 10\hat{k}$$

$$= \langle -2, -6, 10 \rangle$$

$$\vec{v} \times \vec{u} = -1\langle -2, -6, 10 \rangle = \langle 2, 6, -10 \rangle.$$