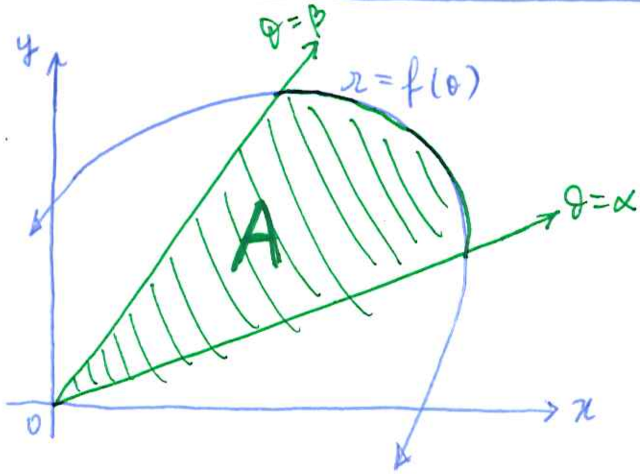
Areas & Lengths in Polar coordinates.

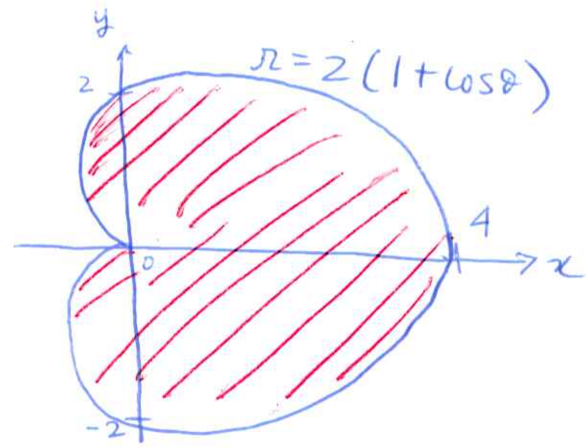


$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

Example ① Cardioid:

$$r = 2(1+\cos\theta)$$

Find the area bounded by the cardioid.



- Determine by graphing, or by using periodicity, that $\theta \in [0, 2\pi]$ sweeps out the entire cardioid exactly once, so:

$$A = \int_0^{2\pi} \frac{1}{2}\left(2(1+\cos\theta)\right)^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(4)(1+2\cos\theta+\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} 2 + 4\cos\theta + 2\cos^2\theta \, d\theta$$

Power reduc'n formula:

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

Example 1, ct'd.

$$A = \int_0^{2\pi} 2 + 4\cos\theta + 2\left(\frac{1}{2}\left(1 + \cos(2\theta)\right)\right) \, d\theta$$

$$= \int_0^{2\pi} 2 + 4\cos\theta + 1 + \cos(2\theta) \, d\theta$$

$$= \int_0^{2\pi} 3 + 4\cos\theta + \cos(2\theta) \, d\theta$$

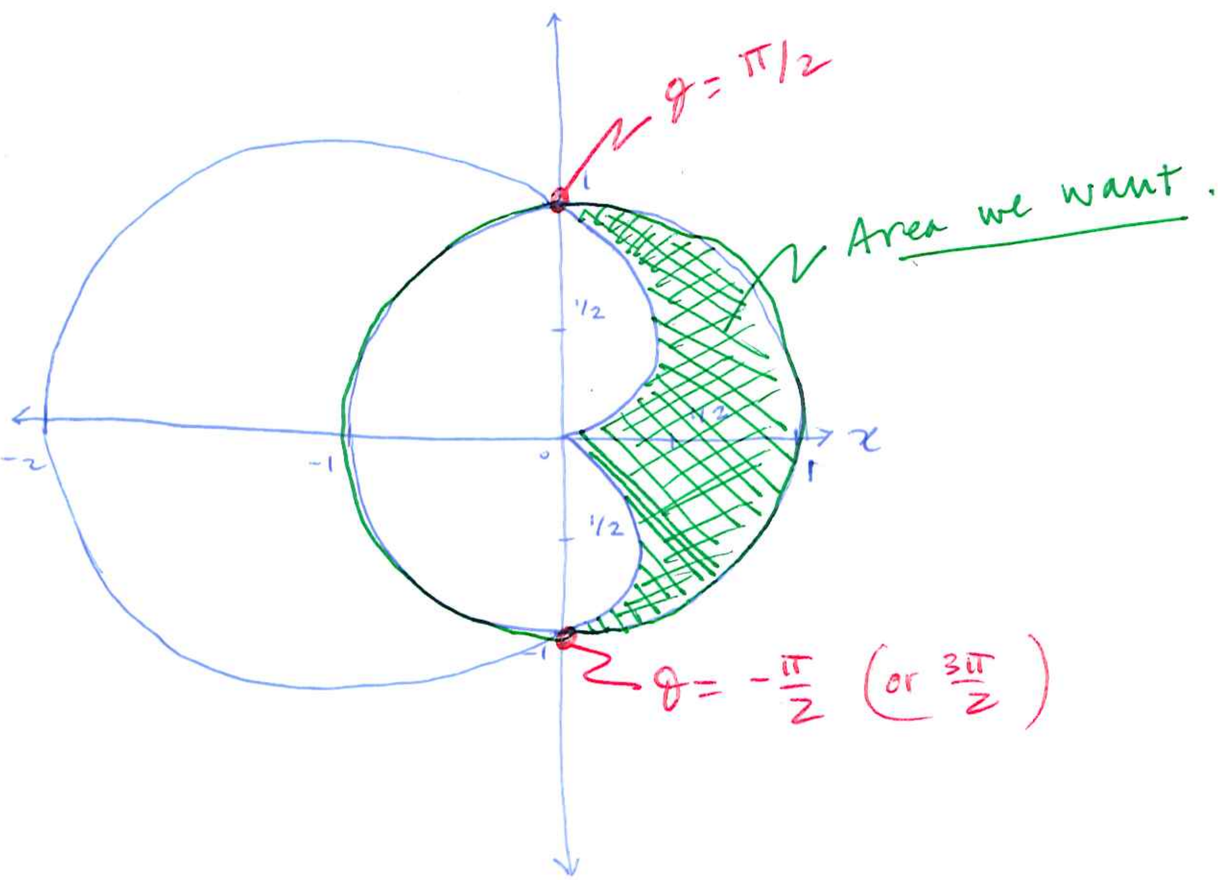
$$= 3\theta + 4\sin\theta + \frac{1}{2}\sin(2\theta) \Big|_0^{2\pi}$$

$$= \left[3(2\pi) + 4\cancelto{0}{\sin(2\pi)} + \frac{1}{2}\cancelto{0}{\sin(2\cdot 2\pi)}\right] - \left[\cancelto{0}{3(0)} + 4\cancelto{0}{\sin(0)} + \frac{1}{2}\cancelto{0}{\sin(2\cdot 0)}\right]$$

$$= \boxed{6\pi}$$

Example ② Find the area of the region tht. lies inside the circle $r=1$ and outside the cardioid $r = 1-\cos\theta$.

SKETCH GRAPH FIRST.



• Compute the area inside the circle, on $[-\pi/2, \pi/2]$,

• Subtract —"— of the cardioid on $[-\pi/2, \pi/2]$.

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1)^2\, d\theta - \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1-\cos\theta)^2\, d\theta$$

Ex. 2, ct'd.

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2} - \frac{1}{2}\left(1 - 2\cos\theta + \cos^2\theta\right)\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cancelto{0}{\frac{1}{2} - \frac{1}{2}} + \cos\theta - \frac{1}{2}\underbrace{\cos^2\theta}_{\cos^2\theta = \frac{1}{2}\left(1 + \cos(2\theta)\right)}\, d\theta$$

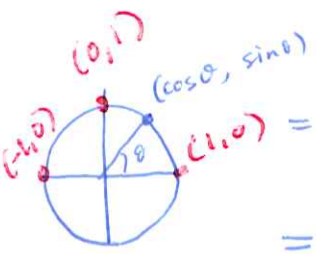
$$= \int_{-\pi/2}^{\pi/2} \cos\theta - \frac{1}{2}\left(\frac{1}{2}\left(1 + \cos(2\theta)\right)\right)\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta - \frac{1}{4} - \frac{1}{4}\cos(2\theta)\, d\theta$$

$$= 2\int_{0}^{\pi/2} \cos\theta - \frac{1}{4} - \frac{1}{4}\cos(2\theta)\, d\theta$$

, as the integrand ~~was~~ is an even fn. of θ, and the interval of integratn was symmetric abt. 0.

$$= 2\left[\sin\theta - \frac{1}{4}\theta - \frac{1}{8}\sin(2\theta)\right]\Big|_{0}^{\pi/2}$$



$$= 2\left[\sin(\pi/2) - \frac{1}{4}(\pi/2) - \frac{1}{8}\sin(2\pi/2)\right] - 2\left[\cancelto{0}{\sin 0} - \frac{1}{4}\cancelto{0}{0} - \frac{1}{8}\cancelto{0}{\sin(2\cdot 0)}\right]$$

$$= 2\left[1 - \frac{\pi}{8} - 0\right] = 2\left[1 - \frac{\pi}{8}\right] = \boxed{2 - \frac{\pi}{4}.}$$