Nov. 16:

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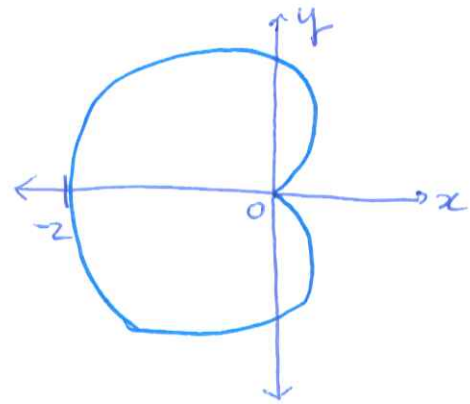
If $r = f(\theta)$ has cts. 1^{st} deriv. for $\alpha \le \theta \le \beta$

and if the point $P(r,\theta)$ traces the curve exactly once as θ runs from α to β,

Then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example ①. Find the length of the cardioid $r = 1 - \cos\theta$.



ALWAYS SKETCH FIRST.

✓ Cardioid is traced out exactly once when θ varies btwn. 0 & 2π.

✓ $\frac{dr}{d\theta} = \frac{d}{d\theta}\left[1 - \cos\theta\right] = \sin\theta$ is cts. for $\theta \in [0, 2\pi]$.

So $$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta =$$

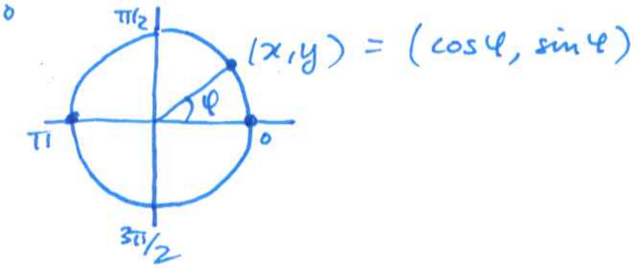
$$= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{=1}}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

Ex. ①, ct'd. $\boxed{\text{Identity: } \sin^2\theta = \frac{1}{2}(1-\cos(2\theta))} \Rightarrow 1-\cos(2\theta) = 2\sin^2\theta$

So $L = \int_0^{2\pi} \sqrt{2}\sqrt{1-\cos\theta}\, d\theta = \int_0^{2\pi} \sqrt{2}\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$

Aside. $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1-\cos\left(2\cdot\frac{\theta}{2}\right)\right) = \frac{1}{2}(1-\cos\theta) \Rightarrow 1-\cos\theta = 2\cdot\sin^2\left(\frac{\theta}{2}\right)$

$= \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta = \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta.$



when $\theta \in [0, 2\pi]$, $\frac{\theta}{2} \in [0, \pi]$.

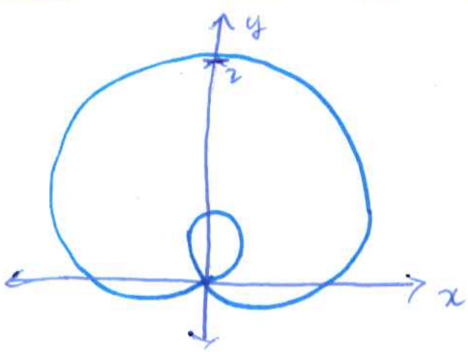
So for $\theta \in [0, 2\pi]$, $\sin\left(\frac{\theta}{2}\right) \geq 0$.

$$L = \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta = -4\cos\left(\frac{\theta}{2}\right)\Big|_0^{2\pi} = -4\cos\left(\frac{2\pi}{2}\right) + 4\cos\left(\frac{0}{2}\right)$$

$$= -4(-1) + 4(1)$$

$$= 4 + 4 = 8.$$

Ex. ② Find the arc length: $r = 1 + 2\sin\theta$



✓ $\theta \in [0, 2\pi]$

✓ $\frac{dr}{d\theta} = 2\cos\theta$

$$L = \int_0^{2\pi} \sqrt{(1+2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 4\sin\theta + 4\sin^2\theta + 4\cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{5 + 4\sin\theta}\, d\theta$$

$$\approx 13.3649$$