Lecture 6: Lesson and Activity Packet

MATH 330: Calculus III

September 23, 2016

Announcements and Homework

- Please turn in Written Homework 2 today if you've finished it
- Canvas Homework due 11:59 p.m. Friday

Recap

- Adding/subtracting finitely many terms
- Reindexing
- Integral test
- Proving harmonic series diverges
- p-series
- Error estimates/remainders of partial sums

Questions on any of this?

If not, then today's lesson will be on the **comparison tests**, and possibly the **root** and **ratio** tests.

...But first, a short recapitulation of what we know so far.

- Definition of partial sums and convergence an infinite series:

For the infinite series $\sum_{n=1}^{\infty} a_n$, the partial sums are defined by

$$S_1 = a_1; \quad S_2 = a_1 + a_2; \quad \cdots \quad S_N = \sum_{n=1}^{N} a_n \quad \cdots,$$

or equivalently by the recursion relation

$$S_1 = a_1; \quad S_n = S_{n-1} + a_n \text{ for } n \geq 2.$$

The series converges if and only if its sequence of partial sums converges, and if there is convergence, then the sum of the series is defined as the limit of the partial sums.

  - Example of why $\sum_{n=1}^{\infty}(-1)^n$ doesn't converge: sequence of partial sums is alternating
  - Example of why $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges: sequence of partial sums converges

- Geometric series:

$\boxed{\sum_{n=0}^{\infty} ar^n \text{ converges to } \frac{a}{1-r} \text{ if } |r| < 1, \text{ and diverges if } |r| \geq 1.}$

  - Dropping a ball example
  - Repeating decimal example

- p-series:

$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1, \text{ and diverges if } p \leq 1.}$

  - Special case: the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Does not converge! Does not converge!
  - For the convergent ones, we don't necessarily know what they converge *to*—that is, we don't know their sums. But we know they converge because of the integral test.

- Telescoping series:

Where partial sums have all but first and "last" terms cancel. For example,

$$\sum_{n=1}^{\infty} \frac{6n+3}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2} \right)$$

has partial sums $S_k = \left(\frac{3}{1^2} - \frac{3}{2^2}\right) + \left(\frac{3}{2^2} - \frac{3}{3^2}\right) + \cdots + \left(\frac{3}{(k-1)^2} - \frac{3}{k^2}\right) = 3 - \frac{3}{k^2}$, which converge to 3, so the series has sum 3. NOTE that to start off this example, we used the partial fraction decomposition.

- Combining series:
  - If $\sum a_n = A$ and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$.
  - If $\sum a_n = A$ then $\sum(ka_n) = kA$.
  - Every nonzero constant multiple of a divergent series diverges too.
  - If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge too.
  - If $\sum a_n$ and $\sum b_n$ *both* diverge, then we cannot say anything about either $\sum(a_n + b_n)$ or $\sum(a_n - b_n)$.
  - Adding or deleting finitely many terms doesn't affect whether a series converges, just (possibly) its sum.
  - Can reindex: $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n-1}$, and in general $\sum_{n=j}^{\infty} a_n = \sum_{n=j+h}^{\infty} a_{n-h}$.
- n^{th} term test for divergence

  If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

  - Converse is false! Counterexample is the harmonic series, whose terms approach zero but which diverges anyway.
- Integral test

  If $a_n = f(n)$ where f is positive, continuous, and decreasing for all $x, n \geq N \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\ dx$ either both converge, or both diverge (sometimes said: "the sum and the integral converge or diverge together").

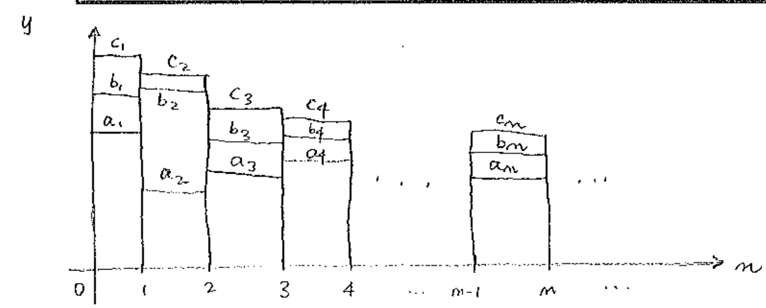
This was just a recap. Hope it helps! But we're not done with series yet. We'd like some more tools to determine whether they converge. One such tool is the **comparison test**.

This test will remind you of the **squeeze theorem** or **sandwich theorem** for functions and sequences. It applies when you have one series you know converges or diverges, and a (similar-looking?) series whose convergence or divergence you don't (yet) know.

Theorem 1 (*Comparison Test*)

Suppose that for all $n > N$, a_n, b_n and c_n are all positive.

- ***If** $\sum c_n$ converges **and** for all $n > N$, $b_n \leq c_n$, **then** $\sum b_n$ converges too.*
- ***If** $\sum a_n$ diverges **and** for all $n > N$, $a_n \leq b_n$, **then** $\sum b_n$ diverges too.*



Key idea: The sum of the series is the same as the sum of the areas of the rectangles (if it exists). So if $\sum c_n$ is finite, that forces $\sum b_n$ to be finite as well, so $\sum b_n$ converges. Likewise, if $\sum a_n$ is infinite, that forces $\sum b_n$ to be infinite too, so $\sum b_n$ diverges.

Group Exercise 1 (*5 minutes*)

What happens if $a_n < b_n$ and $\sum a_n$ converges? Likewise, what happens if $b_n < c_n$ and $\sum c_n$ diverges?

Theorem 1 doesn't tell us about these scenarios.

Example 1

$\sum_{n=1}^{\infty} \frac{5}{5n-1}$. Well, $\frac{5}{5n-1} > \frac{5}{5n} = \frac{1}{n}$, because $5n-1 < 5n$. Also, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So by the comparison test, $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges too.

Example 2

$$\underbrace{5+\frac{2}{3}+\frac{1}{7}+1}_{\text{truncate these}}+\underbrace{\frac{1}{2+1}+\frac{1}{4+\sqrt{2}}+\frac{1}{8+\sqrt{3}}+\frac{1}{16+\sqrt{4}}+\cdots+\frac{1}{2^n+\sqrt{n}}+\cdots}$$

For these, $\frac{1}{2^n+\sqrt{n}} < \frac{1}{2^n}$, because $2^n+\sqrt{n} > 2^n$ for $n \in \mathbb{N}$. Also, $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a geom. series with $a=1$ and $r=\frac{1}{2}$, and as $|a| = \frac{1}{2} < 1$, it converges.

Therefore, by the comparison test, $\sum_{n=0}^{\infty} \frac{1}{2^n+\sqrt{n}}$ converges, and so does our "original"/un-truncated series

More examples of the comparison test will come on the next homework. Now, let's see another test:

Theorem 2 (***Limit comparison test***)
Suppose $a_n > 0$ and $b_n > 0$ for all $n > N$.

1. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then $\sum a_n$ and $\sum b_n$ converge or diverge together.*
2. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$* **and** *$\sum b_n$ converges, then $\sum a_n$ converges too.*
3. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$* **and** *$\sum b_n$ diverges, then $\sum a_n$ diverges too.*

Important note: for the second and third cases, **both** hypotheses must hold!

Group Exercise 2 (***3 minutes***)
If we have $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ diverges, then what happens? If we have $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ converges, then what happens? If we have $\lim_{n\to\infty} \frac{a_n}{b_n} = L \leq 0$, then what happens? [Hint: Your answer for these three questions will be the same.]

The theorem doesn't tell us what happens.

Example 3

$\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ Guess: $\frac{2m+1}{m^2+2m+1} = \frac{2+1/m}{m+2+1/m} \longrightarrow \frac{2}{m+2}$,

as large terms dominate -- so we try to show divergence. Try $b_m := \frac{1}{m}$:

$$\lim_{m\to\infty} \frac{\left(\frac{2m+1}{m^2+2m+1}\right)}{1/m} = \lim_{m\to\infty} \frac{2m^2+m}{m^2+2m+1} = \lim_{m\to\infty} \frac{2+1/m}{1+2/m+1/m^2} = 2.$$

So have case (1), and $\sum_{m=1}^{\infty} \frac{2m+1}{m^2+2m+1}$ and $\sum_{m=1}^{\infty} \frac{1}{m}$ converge or diverge together. Since $\sum_{m=1}^{\infty} \frac{1}{m}$ diverges, so does $\sum_{m=1}^{\infty} 2m+1/m^2+2m+1$ diverges too.

Group Exercise 3 (*optional*)

Determine whether $\sum_{n=1}^{\infty} \frac{\sqrt{x}+3}{x^2-x+1}$ *converges or diverges.*

The order of the terms approaches that of $\frac{\sqrt{m}}{m^2} = m^{-3/2}$.

So try comparing to $\sum_{m=1}^{\infty} \frac{1}{m^{3/2}}$, which converges as a p-series with $p = \frac{3}{2} > 1$.

$$\lim_{m\to\infty} \frac{\left(\frac{\sqrt{m}+3}{m^2-m+1}\right)}{1/m^{3/2}} = \lim_{m\to\infty} \frac{m^2+3m^{3/2}}{m^2-m+1} = \lim_{m\to\infty} \frac{1+3/\sqrt{m}}{1-1/m+1/m^2} = 1.$$

So have Case (1), and because $\sum_{m=1}^{\infty} \frac{1}{m^{3/2}}$ converges, $\sum_{m=1}^{\infty} \frac{\sqrt{m}+3}{m^2-m+1}$ converges too.

Two more tests for convergence/divergence are the **root test** and the **ratio test**. These tests do not require any comparisons to known series, and allow you to work with only the series you are questioning.

Theorem 3 (*Ratio test*)

To determine convergence or divergence of the series $\sum a_n$, *compute* $L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

1. *If* $L < 1$, *then the series converges;*
2. *If* $L > 1$, *then the series diverges;*
3. *If* $L = 1$, *then the test is* ***inconclusive***.

Example 4

$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$

$$L := \lim_{m\to\infty} \left| \frac{\left(\frac{(-1)^{m+1} 2^{m+1}}{(m+1)!}\right)}{\left(\frac{(-1)^m 2^m}{m!}\right)} \right| = \lim_{m\to\infty} \left| \frac{(-1)^{m+1}\, 2^{m+1}\, m!}{(-1)^m\, 2^m\, (m+1)!} \right| =$$

$$= \lim_{m\to\infty} \left| \frac{-1 \cdot 2}{m+1} \right| = 0 < 1,$$ so series converges.

Group Exercise 4 (*7 minutes*)

Does $\sum_{n=1}^{\infty} \frac{n}{2^n}$ *converge or diverge?*

$$L := \lim_{m\to\infty} \left| \frac{(m+1)/2^{m+1}}{m/2^m} \right| = \lim_{m\to\infty} \left| \frac{(m+1)2^m}{m \cdot 2^{m+1}} \right| = \lim_{m\to\infty} \left| \frac{m+1}{2m} \right| =$$

$$= \lim_{m\to\infty} \left| \frac{1+1/m}{2} \right| = \frac{1}{2} < 1,$$ so the series converges.

Theorem 4 (*Root test*)

To determine convergence or divergence of the series $\sum a_n$, compute $\rho := \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

1. *If $\rho < 1$, then the series converges;*
2. *If $\rho > 1$, then the series diverges;*
3. *If $\rho = 1$, then the test is* ***inconclusive****.*

Group Exercise 5 (*10 minutes*)

Does $\sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}}$ converge or diverge? Try the ratio test first.

Ratio: $L := \lim_{n\to\infty} \left| \frac{1/(2^{n+1+(-1)^{n+1}})}{1/2^{n+(-1)^n}} \right| = \lim_{n\to\infty} \left| \frac{2^{n+(-1)^n}}{2^{n+1+(-1)^{n+1}}} \right|$.

If n is odd, then $n+(-1)^n = n-1$ and $n+1+(-1)^{n+1} = n+2$, so $\frac{2^{n+(-1)^n}}{2^{n+1+(-1)^{n+1}}} = \frac{2^{n-1}}{2^{n+2}} = \frac{1}{8}$.

If n is even, then $n+(-1)^n = n+1$ and $n+1+(-1)^{n+1} = n$, so $\frac{2^{n+(-1)^n}}{2^{n+1+(-1)^{n+1}}} = \frac{2^{n+1}}{2^n} = 2$.

The sequence $\left\{ \left| \frac{a_{n+1}}{a_n} \right| \right\}$ therefore doesn't converge, so ratio test inconclusive.

Root: $\rho := \lim_{n\to\infty} \sqrt[n]{\left| 1/2^{n+(-1)^n} \right|} = \lim_{n\to\infty} \left| \frac{1}{2^{n+(-1)^n}} \right|^{1/n} = \lim_{n\to\infty} \left| \frac{1}{2^{1+(-1)^n/n}} \right| = \frac{1}{2} < 1$,

so the series converges.

Recap

- Review of series until now
- Comparison test
- Limit comparison test
- Root test
- Ratio test