30 NOV.

Last time: vector add'n, scalar multiplicat'n.

We have no reason (yet) to believe tht. vector add'n & scalar multiplicat'n have the same nice properties tht. scalar add'n & mult. of 2 scalars have (e.g., commutativity, associativity, etc.) But let's check them:

Let $\vec{u} := \langle u_1, u_2, u_3 \rangle$, $\vec{v} := \langle v_1, v_2, v_3 \rangle$, $\vec{w} := \langle w_1, w_2, w_3 \rangle$, and let $a, b \in \mathbb{R}$.

(1) To prove commutativity, show $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

$$\begin{aligned}
\vec{u} + \vec{v} &= \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle \\
&= \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \\
&= \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle \\
&= \langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle \\
&= \vec{v} + \vec{u}.
\end{aligned}$$

(2) To prove associativity, show $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

(Exercise)

(Other) properties of vector add'n + scalar mult.

(1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ Commutativity

(2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ Associativity

(3) $\vec{u} + \vec{0} = \vec{u}$

(4) $\vec{u} + (-\vec{u}) = \vec{0}$

(5) $0\vec{u} = \vec{0}$

(6) $1\vec{u} = \vec{u}$

(7) $a(b\vec{u}) = (ab)\vec{u}$

(8) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

(9) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

(For any $a, b \in \mathbb{R}$, $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$)

Def. Any vector whose length (magnitude) is 1 is called a unit vector.

Def. "Standard" unit vectors are:

$$\hat{i} := \langle 1, 0, 0 \rangle$$

$$\hat{j} := \langle 0, 1, 0 \rangle$$

$$\hat{k} := \langle 0, 0, 1 \rangle$$

Any vector $\vec{u} := \langle u_1, u_2, u_3 \rangle$ can be written as a linear combination of the standard unit vectors:

$$\vec{u} = \langle u_1, u_2, u_3 \rangle \quad \text{"standard form"}$$

$$= \langle u_1, 0, 0 \rangle + \langle 0, u_2, 0 \rangle + \langle 0, 0, u_3 \rangle$$

$$= u_1 \langle 1, 0, 0 \rangle + u_2 \langle 0, 1, 0 \rangle + u_3 \langle 0, 0, 1 \rangle$$

$$= \underbrace{u_1}_{\text{"i-component"}} \hat{i} + \overbrace{u_2}^{\text{"j-component"}} \hat{j} + \underbrace{u_3}_{\text{"k-component"}} \hat{k}$$

Unit vectors in the direc'n of other (non-unit?) vectors.

Whenever $\vec{v} \neq \vec{0}$, $|\vec{v}| \neq 0$. (Why?),

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

so if we let $\hat{v} := \frac{\vec{v}}{|\vec{v}|}$, then compute $|\hat{v}|$:

$$|\hat{v}| = \left|\frac{\vec{v}}{|\vec{v}|}\right| = \overbrace{\left|\frac{1}{\underbrace{|\vec{v}|}_{\text{mag.}}}\right|}^{\text{abs. val.}} \cdot \underbrace{|\vec{v}|}_{\text{mag.}} = \frac{1}{|\vec{v}|}|\vec{v}| = 1.$$

$$|a\vec{v}| = \underbrace{|a|}_{\text{abs. val.}} \cdot |\vec{v}|$$

So, $\hat{v}$ is the unit vector in the direc'n of $\vec{v}$.

EXAMPLE. Find a unit vector in the same direc'n as the vector from $P(1,0,1)$ to $Q(3,2,0)$.

First, $\vec{PQ} = \langle 3-1, 2-0, 0-1 \rangle = \langle 2, 2, -1 \rangle$.

Then compute magnitude: $|\vec{PQ}| = \sqrt{2^2 + 2^2 + (-1)^2}$
$= \sqrt{4+4+1} = \sqrt{9} = 3$.

$$\hat{PQ} := \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\vec{PQ}}{3} = \frac{1}{3}\langle 2, 2, -1 \rangle = \boxed{\left\langle \frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \right\rangle}.$$