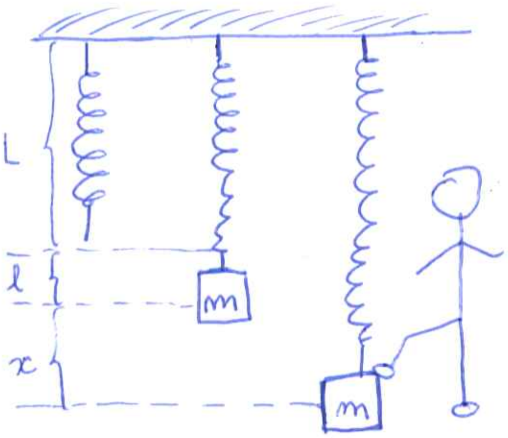
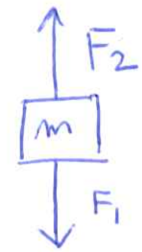
# Nov. 15: Simple Harmonic Motion.




Forces on the mass.




1. Force due to Gravity

   $F_1 = mg$

2. Restoring force of the spring.

Hooke's Law: The magnitude of the force needed to elongate a spring is directly proportional to the amount of elongation (provided small elongation).

$$|F| = k\,s$$

force: $|F|$; "spring constant": $k$; elongation: $s$.

$[s] = m$ (length)

$[F] = N = \frac{kg \cdot m}{s^2}$

$[k] = \frac{kg}{s^2}$.

Example: If a 30-lb. weight stretches a spring by 2 ft. (from the spring's equilibrium):

$$30 \text{ lb} = k \cdot 2 \text{ ft}.$$

$$k = \frac{30 \text{ lb}}{2 \text{ ft}} = 15 \frac{\text{lb}}{\text{ft}}.$$

By Hooke's law, $F_2 = -k(\ell + x)$.

Consider the spring-mass system in equilibrium:

$$F_1 = F_2, \text{ i.e., } -mg = -k\ell \Rightarrow k\ell = mg.$$

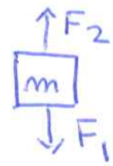
So, because $kl = mg$, we can express $F_2$ as

$$F_2 = -k(l+x) = -kl - kx = -mg - kx.$$

So our forces on the mass are:

$$F_1 = mg$$

$$F_2 = -mg - kx(t)$$




By Newton's $2^{nd}$ Law, if $F := F_1 + F_2$, then the mass has acceleration $a(t)$ given by $F = m\,a(t)$.

For our spring-mass system,

$$F = F_1 + F_2 = mg + (-mg - kx(t))$$

$$F = -kx(t),$$

and so $-kx(t) = m\,a(t)$. That is,

$$m\frac{d^2x}{dt^2} + kx(t) = 0.$$

A $2^{nd}$-order linear ODE; by our Theorem, there are 2 (& only 2) linearly independent sol'ns $x_1(t)$ and $x_2(t)$, and the general sol'n to the ODE is $x(t) := c_1 x_1(t) + c_2 x_2(t)$.

Furthermore, the ODE is a constant-coefficient one, so we can find those sol'ns.

$$m \frac{d^2x}{dt^2} + kx(t) = 0.$$

Assume sol'n of the form $x(t) = e^{pt}$, substitute into ODE:

$$e^{pt}\left[mp^2 + k\right] = 0,$$

so characteristic eq'n is $mp^2 + k = 0$, and is solved
$p^2 + \frac{k}{m} = 0$
by $p = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}}$.

Cplx roots of the characteristic eq'n imply tht. the general sol'n is

$$x(t) = c_1 \sin\left(t\sqrt{\frac{k}{m}}\right) + c_2 \cos\left(t\sqrt{\frac{k}{m}}\right).$$

Suppose that the motion starts, at $t=0$, with the mass at rest at position $x_0$.

$$x(0) = x_0$$
$$v(0) = x'(0) = 0.$$

By the general sol'n to the ODE, $x(0) = c_1 \sin(0) + c_2\cos(0)$
$x(0) = c_2$.

So, applying the IC $x(0) = x_0$, we get $c_2 = x_0$.

By the gen. sol'n, $x'(t) = c_1\sqrt{\frac{k}{m}}\cos\left(t\sqrt{\frac{k}{m}}\right) - \overset{x_0}{c_2}\sqrt{\frac{k}{m}}\sin\left(t\sqrt{\frac{k}{m}}\right)$

so $x'(0) = c_1\sqrt{\frac{k}{m}}$. By the IC, $c_1 = 0$.

So the sol'n to the initial value problem

$$\begin{cases} m \dfrac{d^2x}{dt^2} + k\,x(t) = 0 \\ x(0) = x_0 \ , \quad x'(0) = 0 \end{cases}$$

is $$x(t) = x_0 \cos\left(t\sqrt{\tfrac{k}{m}}\right).$$

A quick modification is to assume a nonzero initial velocity:

$$\begin{cases} m \dfrac{d^2x}{dt^2} + k\,x(t) = 0 \\ x(0) = x_0 \ , \quad x'(0) = v_0 \ , \quad v_0 \neq 0 \ . \end{cases}$$

Same gen. sol'n $x(t) = c_1 \sin\left(t\sqrt{\frac{k}{m}}\right) + c_2 \cos\left(t\sqrt{\frac{k}{m}}\right)$.

Same val. for $c_2$ : $x(0) = x_0 = c_2$ , so

$$x(t) = c_1 \sin\left(t\sqrt{\tfrac{k}{m}}\right) + x_0 \cos\left(t\sqrt{\tfrac{k}{m}}\right).$$

So $x'(t) = c_1\sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}}\, t\right) - x_0\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

$x'(0) = c_1\sqrt{\frac{k}{m}} = v_0$ , by the I.C. — so,

we obtain $c_1 = v_0\sqrt{\frac{m}{k}}$ . $= \dfrac{v_0}{\sqrt{k/m}} = \dfrac{v_0}{(\sqrt{k}/\sqrt{m})} = \left(\dfrac{\sqrt{m}}{\sqrt{k}}\right) v_0$

Therefore, the sol'n to the IVP is

$$x(t) = v_0\sqrt{\tfrac{m}{k}}\, \sin\left(\sqrt{\tfrac{k}{m}}\, t\right) + x_0 \cos\left(\sqrt{\tfrac{k}{m}}\, t\right).$$

<u>Notice</u>: This sol'n is <u>oscillatory</u> — but in real life, the mass stops bouncing eventually. We've left out <u>damping</u>.

Forces on the man.

1. $F_1 = mg$
2. $F_2 = -mg - kx$
3. Damping force.

Resistance of the medium (air) is not known exactly, but can be approximated as proportional to the magnitude of the velocity. i.e.,

$$|F_3| = a\left|\frac{dx}{dt}\right| \quad , \text{ where } a > 0 \text{ is the damping constant.}$$

$$F_3 = -a\frac{dx}{dt}$$

[Aside: $[F_3] = N = \frac{kg \cdot m}{s^2}$

$\left[\frac{dx}{dt}\right] = \frac{m}{s}$ ,

so $[a] = \frac{kg}{s}$ .]

Applying Newton's 2nd law to $F = F_1 + F_2 + F_3 = -kx - a\frac{dx}{dt}$,

we obtain $\underbrace{F = m\frac{d^2x}{dt^2} = -kx - a\frac{dx}{dt}}$ , i.e.,

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0.$$

Roots of char. eq'n are $p = \dfrac{-a \pm \sqrt{a^2 - 4mk}}{2m}$ .

Depending on the discriminant $a^2 - 4mk$, may have sol'ns $x(t) = e^{-\frac{a}{2m}t}\left[c_1 \sin\left(\frac{\sqrt{a^2-4mk}}{2m}t\right) + c_2 \cos\left(\ \cdot t\right)\right]$