Nov. 17: Reduction of Order (23A,B).

We know how to solve $a_m y^{(m)} + a_{m-1} y^{(m-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = Q(x)$

m^{th} ord., lin., const. coeff. eq'n.

We had a theorem that guaranteed existence & uniqueness of sol'ns to the general n^{th} order linear ODE:

$$(*) \qquad f_m(x) \frac{d^m y}{dx^m} + f_{m-1}(x) \frac{d^{m-1} y}{dx^{m-1}} + \cdots + f_1(x) \frac{dy}{dx} + f_0(x) y = Q(x),$$

but — in general — these sol'ns are difficult, if not impossible, to find.

In the VERY SPECIAL CASE when we know/are given $n-1$ many linearly independent sol'ns of $(*)$, we can use reduc'n of order to find the n^{th} sol'n (lin. indep. from others).

Start with 2^{nd} order general case:

$$f_2(x) y''(x) + f_1(x) y'(x) + f_0(x) y(x) = Q(x)$$

or homog. version: $f_2(x) y'' + f_1 y' + f_0 y = 0$.

Assume $y_1(x)$ solves the homogeneous eq'n.

Assume $y_2(x)$ has the form

$$y_2(x) = y_1(x) \int u(x)\, dx,$$ for some unknown function u.

First, we seek y_2 (that is, want to find u) s.t. y_2 satisfies the homog. eq'n, & is l.i. from y_1.

Differentiating: $y_2' = y_1' \int u \, dx + y_1 u$

$$y_2'' = y_1'' \int u \, dx + y_1' u + y_1' u + y_1 u'$$

$$= y_1'' \int u \, dx + 2y_1' u + y_1 u'.$$

Subst. into homog. ODE:

$$f_2 y_2'' + f_1 y_2' + f_0 y_2 = f_2\left[y_1'' \int u \, dx + 2y_1' u + y_1 u'\right] + f_1\left[y_1' \int u \, dx + y_1 u\right] + f_0 y_1 \int u \, dx = 0$$

$$\underbrace{\left[f_2 y_1'' + f_1 y_1' + f_0 y_1\right]}_{=0,\text{ as } y_1 \text{ solved homog. eq'n.}} \int u \, dx + \left[2f_2 y_1' + f_1 y_1\right] u + f_2 y_1 u' = 0$$

$$2f_2 y_1' u + f_1 y_1 u + f_2 y_1 u' = 0$$

div. both sides by $u \cdot y_1 \cdot f_2$ $\longrightarrow$ $f_2 y_1 \frac{du}{dx} + 2 f_2 u \frac{dy_1}{dx} = -f_1 y_1 u$

$$\frac{1}{u}\frac{du}{dx} + \frac{2}{y_1}\frac{dy_1}{dx} = \frac{-f_1}{f_2}$$

$$\int \left[\frac{1}{u}\frac{du}{dx} + \frac{2}{y_1}\frac{dy_1}{dx}\right] dx = \int \frac{-f_1}{f_2} dx$$

$$\int \frac{1}{u}\frac{du}{dx} dx + \int \frac{2}{y_1}\frac{dy_1}{dx} dx = \int \frac{-f_1}{f_2} dx$$

let $w := u$, $\frac{dw}{dx} = \frac{du}{dx}$, $dw = \frac{du}{dx} dx$ | let $v := y_1$, $dv = \frac{dy_1}{dx} dx$

$$\int \frac{1}{w} dw + \int \frac{2}{v} dv = \int \frac{-f_1}{f_2} dx$$

$$\int \frac{1}{u} du + \int \frac{2}{y_1} dy_1 = \int -\frac{f_1}{f_2} dx.$$

$$\ln|u| + 2\ln|y_1| = \int -\frac{f_1}{f_2} dx$$

$\ln(a) + \ln(b) = \ln(ab)$
$c \cdot \ln(a) = \ln(a^c)$

$$\ln\left(|u| y_1^2\right) = \int -\frac{f_1}{f_2}\, dx$$

$$|u| y_1^2 = \exp\left(\int -\frac{f_1}{f_2}\, dx\right)$$

$$|u| = \frac{1}{y_1^2} \exp\left(\int \frac{-f_1}{f_2}\, dx\right)$$

all consts. of integratin are assumed 0

$$u = \frac{1}{y_1^2} \exp\left(\int -\frac{f_1}{f_2}\, dx\right).$$

and abs. value signs don't matter, as we sought any u that made $y_2 = y_1 \int u\, dx$ satisfy the homog. ODE.

Need to prove that y_1 & y_2 are linearly independent.

Reminder: $$y_2 = y_1 \int \frac{1}{y_1^2} \exp\left(\int -\frac{f_1}{f_2}\, dx\right) dx$$

$$y_2' = y_1' \int \frac{1}{y_1^2} \exp\left(\int -\frac{f_1}{f_2}\, dx\right) dx + y_1\left(\frac{1}{y_1^2} \exp\left(\int -\frac{f_1}{f_2}\, dx\right)\right)$$

$$= y_1' \int \frac{1}{y_1^2} \exp\left(\int -\frac{f_1}{f_2}\, dx\right) dx + \frac{1}{y_1} \exp\left(\int -\frac{f_1}{f_2}\, dx\right)$$

To prove $\{y_1, y_2\}$ is a l.i. set of functions, use the Wronskian:

$$W(\{y_1, y_2\}) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_1 \int \frac{1}{y_1^2} \exp\left(\int -\frac{f_1}{f_2}\, dx\right) dx \\ y_1' & y_1' \int \frac{1}{y_1^2} \exp\left(\int -\frac{f_1}{f_2}\, dx\right) dx + \frac{1}{y_1} \exp\left(\int -\frac{f_1}{f_2}\, dx\right) \end{vmatrix}$$

$$W(\{y_1, y_2\}) = y_1\left(y_1' \int \frac{1}{y_1^2} \exp\left(\int -\frac{f_1}{f_2}\,dx\right)dx + \frac{1}{y_1}\exp\left(\int -\frac{f_1}{f_2}\,dx\right)\right) -$$

$$- y_1'\left(y_1 \int \frac{1}{y_1^2} \exp\left(\int -\frac{f_1}{f_2}\,dx\right)dx\right)$$

$$= \exp\left(\int -\frac{f_1}{f_2}\,dx\right) \neq 0 \quad \text{for any } x \in \mathbb{R},$$

so — indeed — $\{y_1, y_2\}$ is lin. indep.

We've shown that, given the ODE $f_2 y'' + f_1 y' + f_0 y = Q(x)$, and given a sol'n, y_1, to the homog. vrsn. $f_2 y'' + f_1 y' + f_0 y = 0$,

$$y_2 := y_1 \int \frac{1}{y_1^2} \exp\left(\int -\frac{f_1}{f_2}\,dx\right)dx$$ is a 2^{nd} ~~lin. indep.~~ sol'n to the homog. eq'n, that's lin. indep. from y_1.

Let's now seek the particular sol'n y_p to the non-homog. ODE.

Guess: $y_p := y_1 \int u\,dx$. Subst. into the non-homog. ODE:

$$2 f_2 y_1' u + f_1 y_1 u + f_2 y_1 u' = Q(x).$$

$$u' + \left[\frac{2y_1'}{y_1} + \frac{f_1}{f_2}\right]u = \frac{Q}{f_2 y_1}.$$ Int. factor:

Let $\mu(x) := \exp\left(\int \frac{2y_1'}{y_1} + \frac{f_1}{f_2}\,dx\right) = \exp\left(\int \frac{2}{y_1}\frac{dy_1}{dx}\,dx + \int \frac{f_1}{f_2}\,dx\right) =$

$$= \exp\left(\int \frac{2}{y_1}\,dy + \int \frac{f_1}{f_2}\,dx\right) = \exp\left(2\ln|y_1| + \int \frac{f_1}{f_2}\,dx\right) =$$

$$\boxed{e^{a+b} = e^a e^b}$$

$$= \exp\left(\ln(y_1^2) + \int \frac{f_1}{f_2}\,dx\right) = y_1^2 \exp\left(\int \frac{f_1}{f_2}\,dx\right).$$

Check this step! (Verify the int. factor.)

$$\underbrace{y_1^2 \exp\left(\int \frac{f_1}{f_2}\,dx\right) u' + y_1^2 \exp\left(\int \frac{f_1}{f_2}\,dx\right)\left[\frac{2y_1'}{y_1} + \frac{f_1}{f_2}\right] u} = \frac{y_1^2 \exp\left(\int \frac{f_1}{f_2}\,dx\right) Q}{f_2\, y_1}$$

$$\frac{d}{dx}\left[y_1^2 \exp\left(\int \frac{f_1}{f_2}\,dx\right) u\right] = \frac{y_1 Q}{f_2} \exp\left(\int \frac{f_1}{f_2}\,dx\right)$$

$$y_1^2 \exp\left(\int \frac{f_1}{f_2}\,dx\right) u = \int \frac{y_1 Q}{f_2} \exp\left(\int \frac{f_1}{f_2}\,dx\right) dx$$

$$u = \frac{1}{y_1^2} \exp\left(-\int \frac{f_1}{f_2}\,dx\right) \int \frac{y_1 Q}{f_2} \exp\left(\int \frac{f_1}{f_2}\,dx\right) dx .$$

So $$y_p = y_1 \int \left(\frac{1}{y_1^2} \exp\left(-\int \frac{f_1}{f_2}\,dx\right) \int \frac{y_1 Q}{f_2} \exp\left(\int \frac{f_1}{f_2}\,dx\right) dx\right) dx .$$

EXAMPLE. $\underset{f_2}{\underline{(x^2+1)}}\, y'' \underset{f_1}{\underline{-2x}}\, y' + 2y = 0$ has a solution $\underset{y_1}{\underline{y_1 = x}}$. Find a 2nd sol'n y_2 to the homog. ODE that's l.i. from y_1.

Verify: $y_1 = x$ solves the ODE? ✓ $y_1' = 1$, $y_1'' = 0$.

Subst.: $(x^2+1)y_1'' - 2xy_1' + 2y_1 = \cancel{(x^2+1)(0)}^{\,0} - 2x(1) + 2(x)$
$= -2x + 2x = 0$ ✓.

$$y_2 = y_1 \int \frac{1}{y_1^2} \exp\left(\int \frac{-f_1}{f_2}\,dx\right) dx = x \int \frac{1}{x^2} \exp\left(\int \frac{-(-2x)}{x^2+1}\,dx\right) dx =$$

$$= x \int \frac{1}{x^2} \exp\left(\int \frac{2x}{x^2+1}\,dx\right) dx = x \int \frac{1}{x^2} \exp\left(\int \frac{1}{w}\,dw\right) dx = x \int \frac{1}{x^2} \exp(\ln|w|)\,d$$

Let $w = x^2+1$
$dw = 2x\,dx$

$$= x \int \frac{1}{x^2} |w|\,dx = x \int \frac{1}{x^2} |x^2+1|\,dx = x \int \frac{x^2+1}{x^2}\,dx = x \int 1 + \frac{1}{x^2}\,dx$$

$$y_2 = x\int 1 + \frac{1}{x^2}\,dx = x\left[x - \frac{1}{x}\right] = x^2 - \frac{x}{x} = x^2 - 1.$$

Verify y_2 solves the homog. ODE:

$y_2' = 2x$, $y_2'' = 2$, so substituting to LHS of ODE:

$$(x^2+1)(2) - 2x(2x) + 2(x^2-1) = 2x^2 + 2 - 4x^2 + 2x^2 - 2$$
$$= \cancelto{0}{(4-4)}\,x^2 + \cancelto{0}{(2-2)} = 0 \quad \checkmark.$$